

NOTES ON GENUS ONE REAL GROMOV-WITTEN INVARIANTS

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ABSTRACT. In this paper, we propose a definition of genus one real Gromov-Witten invariants for certain symplectic manifolds with real a structure, including Calabi-Yau threefolds, and use equivariant localization to calculate certain genus 1 real invariants of the projective space. For this definition, we combine three moduli spaces corresponding to three possible types of involutions on a symplectic torus, by gluing them along common boundaries, to get a moduli space without codimension-one boundary and then study orientation of the total space. Modulo a technical conjectural lemma, we can prove that the result is an invariant of the corresponding real symplectic manifold. In the aforementioned example, our main motivation is to show that the physicists expectation for the existence of separate Annulus, Mobius, and Klein bottle invariants may not always be true.

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1. INTRODUCTION

In this paper, (X, ω, ϕ) will be a symplectic manifold with a real structure ϕ (an antisymplectic involution), i.e. a diffeomorphism $\phi: X \rightarrow X$ such that $\phi^2 = \text{id}_X$ and $\phi^*\omega = -\omega$. Let $L = \text{Fix}(\phi) \subset X$ be the fixed point locus of ϕ ; L is a Lagrangian submanifold of (X, ω) which can be empty. An almost complex structure J on TX is called (ω, ϕ) -**compatible** if $\phi^*J = -J$ and $\omega(\cdot, J\cdot)$ is a metric. Denote by \mathcal{J}_ϕ to be the space of (ω, ϕ) -compatible almost complex structures.

Let $(\Sigma, \omega_\Sigma, \sigma)$ be a real two dimensional symplectic manifold and $j \in \mathcal{J}_\sigma$ be a (ω_Σ, σ) -compatible complex structure (every almost complex structure in real dimension two is integrable). For simplicity, we drop ω_Σ from the notation and call a tuple (Σ, σ, j) a **real curve**. Two antisymplectic involutions, σ_1, σ_2 , on (Σ_1, ω_1) and (Σ_2, ω_2) are said to be equivalent if there is a symplectic diffeomorphism

$$\psi: (\Sigma_1, \omega_1) \rightarrow (\Sigma_2, r\omega_2) \quad \text{for some } r \in \mathbb{R}^*,$$

such that $\sigma_1 = \psi^{-1} \circ \sigma_2 \circ \psi$. For all $g \geq 0$, there are $\left\lfloor \frac{3g+4}{2} \right\rfloor$ equivalence classes of involutions on a topological surface of genus g ; see [15].

Fix ϕ , an equivalence class of σ and $J \in \mathcal{J}_\phi$. For $g, k, \ell \in \mathbb{Z}^{\geq 0}$ and $A \in H_2(X)$, define $\mathcal{M}_{g,k,\ell}(X, A)^{\phi, \sigma}$ to be the moduli space of degree A genus g (ϕ, σ) -real J -holomorphic maps

$$u: \Sigma \rightarrow X, \quad du + J \circ du \circ j = 0, \quad \phi \circ u = u \circ \sigma,$$

where (Σ, σ, j) is a genus g real curve with ℓ disjoint ordered conjugate pairs of marked points, $(z_i, \sigma(z_i))_{i=1}^\ell$, away from $\text{Fix}(\sigma)$, along with k real (σ -fixed) marked points, $(w_i)_{i=1}^k$, if $\text{Fix}(\sigma) \neq \emptyset$, and $A = [u(\Sigma)] \in H_2(X)$ describes the homology class of the image.

Two such tuples, $(u_1, \sigma, j_1, (w_i^1), (z_i^1, \sigma(z_i^1)))$ and $(u_2, \sigma, j_2, (w_i^2), (z_i^2, \sigma(z_i^2)))$, are said to be equivalent to each other if there is a real biholomorphic map $h: (\Sigma, j_1) \rightarrow (\Sigma, j_2)$,

$$h \circ \sigma = \sigma \circ h, \quad u_2 = u_1 \circ h, \quad w_i^2 = h(w_i^1), \quad z_i^2 = h(z_i^1).$$

We denote by $G_{\sigma, j}((w_i)_{i=1}^k, (z_i, \sigma(z_i))_{i=1}^\ell)$ to be the space of real biholomorphic automorphisms of $(u, \sigma, j, (w_i), (z_i, \sigma(z_i)))$ and by $\mathcal{M}_{\sigma, k, \ell}$ to be the space of (smooth) (k, ℓ) -pointed complex structures $j \in \mathcal{J}_\sigma$ modulo real biholomorphic automorphisms. If $k, \ell = 0$, we simply write \mathcal{M}_σ instead of $\mathcal{M}_{\sigma, 0, 0}$. For $g = 1$, \mathcal{M}_σ is a one dimensional manifold, and for $g > 1$, it is an orbifold of real dimension $3g - 3$.

By [11, Appendix C]

$$(1.1) \quad \dim_{\mathbb{R}}^{\text{vir}} \mathcal{M}_{g,k,\ell}(X, A)^{\phi, \sigma} = c_1^{TX}(A) + (\dim_{\mathbb{C}} X - 3)(1 - g) + k + 2\ell.$$

Let $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma}$ be the stable compactification of $\mathcal{M}_{g,k,\ell}(X, A)^{\phi, \sigma}$ (assuming $g + k + 2\ell \geq 3$ if $A = 0$). Let

$$(1.2) \quad \begin{aligned} \text{ev}_i^{\mathbb{R}}: \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma} &\rightarrow L, & \text{ev}_i^{\mathbb{R}}([u, j, (w_a)_{a=1}^k, (z_b, \sigma(z_b))_{b=1}^\ell]) &= u(w_i), \\ \text{ev}_i: \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma} &\rightarrow X, & \text{ev}_i([u, j, (w_a)_{a=1}^k, (z_b, \sigma(z_b))_{b=1}^\ell]) &= u(z_i), \end{aligned}$$

be the natural evaluation maps.

For the classical moduli space $\overline{\mathcal{M}}_{g,n}(X, A)$ of degree A genus g J -holomorphic maps, **Gromov-Witten invariants** are defined via integrals of the form

$$(1.3) \quad \langle \theta_1, \dots, \theta_n \rangle_{g,A} = \int_{[\overline{\mathcal{M}}_{g,n}(X,A)]^{\text{vir}}} \text{ev}_1^*(\theta_1) \wedge \dots \wedge \text{ev}_n^*(\theta_n),$$

where θ_i 's are cohomology classes on X ; see [3],[9],[14]. These integrals make sense and are independent of J , because $\overline{\mathcal{M}}_n(X, A)$ has a virtually orientable fundamental cycle without real codimension one boundary. One would like to define similar invariants for the moduli spaces $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma}$ and the evaluation maps in (1.2). The existence of such invariants is predicted by physicists (see [17] for a genus one example), but there are obstacles to defining such invariants mathematically. In addition to the transversality issues (which are also present in the classical case), issues concerning orientability and codimension-one boundary arise.

In [2], we defined genus 0 real invariants with no real marked points. In this paper, we extend those ideas to the genus 1 case, still with no real marked points.

For every genus one real curve (Σ, σ, i) , define $\text{ind}(\sigma) \geq 0$ to be the number of components in $\text{Fix}(\sigma)$. There are three equivalence classes of antisymplectic involutions on the symplectic torus $T = (\mathbb{R}^2/\mathbb{Z}^2, dx \wedge dy)$, distinguished by their index or by topological types of their quotients. These are

$$(1.4) \quad \begin{aligned} \mathbf{c}_a: (x, y) &\rightarrow (x, -y), & \text{ind}(\mathbf{c}_a) &= 2, & T/\mathbf{c}_a &\cong \text{Annulus}; \\ \mathbf{c}_m: (x, y) &\rightarrow (x + \frac{1}{2}, -y), & \text{ind}(\mathbf{c}_m) &= 1, & T/\mathbf{c}_m &\cong \text{Möbius band}; \\ \mathbf{c}_k: (x, y) &\rightarrow (x + y, -y), & \text{ind}(\mathbf{c}_k) &= 0, & T/\mathbf{c}_k &\cong \text{Klein bottle}. \end{aligned}$$

In the genus zero case, there are two equivalence classes of involutions. Similar to [2, Page 2], let τ and η denote the possible involutions on \mathbb{P}^1 with and without fixed points, respectively. In projective coordinates they are given by

$$\begin{aligned} \tau: [x, y] &: [\bar{y}, \bar{x}], & \text{Fix}(\tau) &\cong S^1, \\ \eta: [x, y] &: [-\bar{y}, \bar{x}], & \text{Fix}(\eta) &\cong \emptyset. \end{aligned}$$

Whereas classical moduli spaces of closed curves have a canonical orientation induced by J , $\mathcal{M}_{g,k,\ell}(X, A)^{\phi,\sigma}$ is not necessarily orientable. Moreover, if it is orientable, there is no canonical orientation. A **real structure** on a vector bundle $E \rightarrow X$ is an anticomplex linear involution $\phi_E: E \rightarrow E$ covering ϕ . A **real square root** of a complex line bundle $E \rightarrow X$ with real structure ϕ_E is a complex line bundle $E' \rightarrow X$ with real structure $\phi_{E'}$ such that

$$(E, \phi_E) \cong (E' \otimes E', \phi_{E'} \otimes \phi_{E'}).$$

The involution ϕ on X canonically lifts to an involution ϕ_{K_X} on the complex line bundle $K_X = \Lambda_{\mathbb{C}}^{\text{top}} T^*X$.

Theorem 1.1. *Let (X^{2n}, ω, ϕ) be a real $2n$ -dimensional symplectic manifold with a real structure. If n is odd, $4|c_1(TX)$, (K_X, ϕ_{K_X}) admits a real square root, and $L = \text{Fix}(\phi)$ is spin, all moduli spaces $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma}$ are orientable. Moreover, a choice of a spin structure on L and a real isomorphism $(K_X, \phi_{K_X}) \cong (E \otimes E, \phi_E \otimes \phi_E)$ canonically determines the orientation.*

This theorem is proved in Section 5. In [2, Prop 1.5], two sufficient conditions for the existence of real square root is provided.

Remark 1.2. These notes, in its current form, was written about a year ago and the results were presented in Simons center, June 3rd 2013. I was expecting to prove the conjectural Lemma 6.3 and complete the construction, but a proof of this lemma seems to demand more than what we know about real curves at this time. Meanwhile, in [7], P. Georgieva and A. Zinger approached the orientability problem for individual components, for all $g \geq 1$, from a much broader and somewhat different point of view. Their orientability results cover a larger class of targets. As far as I understand, they are intending to prove the existence of individual Annulus, Mobius, and Klein bottle invariants, at least for Calabi-Yau manifolds. I am thankful to both of them for sharing their insights and comments. I welcome any comment or suggestion on the conjectural Lemma 6.3.

Each $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma}$ might have several real codimension-one boundary components. Boundary of $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma}$ includes real J -holomorphic maps

$$u: (\Sigma_{\text{nod}}, \sigma_{\text{nod}}, j_{\text{nod}}) \rightarrow (X, \phi, J),$$

defined over real nodal curves. Those in $\partial^1 \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma}$ (virtually) correspond to 1-nodal real curves such that the unique node q lies in $\text{Fix}(\sigma_{\text{nod}})$. In Section 3, we observe that each of these boundary terms is either a common boundary between two of moduli spaces $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma_1}$ and $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma_2}$, $\sigma_1 \neq \sigma_2$, or is indeed a hyper-surface within one of the moduli spaces $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma}$. Therefore, after identifying the common boundary terms, the union of the moduli spaces,

$$\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi} = \coprod_{\sigma} \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi,\sigma},$$

(virtually) has no real codimension one boundary.

If the complex dimension is odd, $4|c_1(TX)$, L is spin, and K_X has a real square root, by Theorem 1.1 we know that every $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi,\sigma}$ is orientable. By studying the orientation along all the common boundary components, we would like to eventually show that the union of them is also orientable.

Conjecture 1.3. *If (X^{2n}, ω, ϕ) is a symplectic manifold with a real structure ϕ , then $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi}$ has a topology with respect to which it is compact and Hausdorff. It has a Kuranishi structure without boundary of virtual real dimension*

$$d = c_1(A) + 2\ell$$

and thus determines an element of $H_d(\overline{\mathcal{M}}_{1,0,\ell}(X, A)^\phi, \mathcal{O})$, where \mathcal{O} is the orientation bundle. If in addition n is odd, $4|c_1(TX)$, K_X has a real square root, and L is spin, then $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^\phi$ is orientable and determines an element of $H_d(\overline{\mathcal{M}}_{1,0,\ell}(X, A)^\phi, \mathbb{Q})$.

We prove the first part of this Conjecture in Section 3. Modulo conjectural Lemma 6.3, we can also prove the second part; see Section 6. In that case for $(\theta_i)_{i=1}^\ell \in H^*(X)$ and $A \in H_2(X)$, we define **real genus 1** Gromov-Witten invariants of X to be

$$\langle \theta_1, \dots, \theta_\ell \rangle_{1,A}^{\text{real}} = \int_{[\overline{\mathcal{M}}_{1,0,\ell}(X, A)^\phi]^{\text{vir}}} \text{ev}_1^* \theta_1 \wedge \dots \wedge \text{ev}_\ell^* \theta_\ell.$$

The moduli space $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^\phi$ provides a framework to define genus one real GW invariants without any restriction on the topology of the image or the involution.

Of particular interest is the case where $n = 3$, $c_1^{TX} \equiv 0$, and $\ell = 0$. In this case

$$\dim^{\text{vir}} \overline{\mathcal{M}}_{1,0,0}(X, A)^\phi = 0$$

and we define

$$N_1^\phi(A) = \#[\overline{\mathcal{M}}_{1,0,0}(X, A)^\phi]^{\text{vir}} \in \mathbb{Q}.$$

These numbers will be independent of the choice of the almost complex structure J (and other choices). If X is a complete intersection Calabi-Yau threefold in a projective space, using equivariant localization technique, the numbers $N_1^\phi(d)$, where $d \in H_2(X) \cong \mathbb{Z}$ is the degree and ϕ is the restriction of complex conjugation to X , are calculated in [17] (assuming they are defined mathematically). These calculations are verified to be weight independent in [13]. Moreover, in [13], the contribution of each of three types of involution is weight independent. In Example 7.4, we show that this does not always happen and only sum of these three numbers is a mathematically well-defined invariant. One might be able to prove the existence of separate invariants for each type of involution on a Calabi-Yau target space, but our technique does not predict the existence of finer invariants.

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2. REAL STRUCTURES ON THE TORUS

Let $(T, \omega_T) = (\mathbb{R}^2/\mathbb{Z}^2, dx \wedge dy)$ be the symplectic torus of volume one. Up to scaling of the volume, any other symplectic tori is symplectomorphic to (T, ω_T) . There are three equivalence classes of antisymplectic involutions on (T, ω_T) , given by (1.4). For each choice of σ on (T, ω_T) , \mathcal{M}_σ is a real one-dimensional space. For $\sigma = \mathbf{c}_a$,

$$\mathcal{M}_{\mathbf{c}_a} = \{\mathbb{C}/\mathbb{Z} + i\mathbb{Z} | b \in \mathbb{R}^{>0}\}.$$

Remark 2.1. Two complex structures on T which represent the same Riemann surface may correspond to different points in \mathcal{M}_σ . For example $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ and $\mathbb{C}/\mathbb{Z} + \frac{i}{b}\mathbb{Z}$ are biholomorphic to each other, but there is no biholomorphic map between $(\mathbb{C}/\mathbb{Z} + i\mathbb{Z}, \mathbf{c}_a)$ and $(\mathbb{C}/\mathbb{Z} + \frac{i}{b}\mathbb{Z}, \mathbf{c}_a)$ which is real with respect to $\mathbf{c}_a(z) = \bar{z}$.

Similarly, for $\sigma = \mathfrak{c}_m$ and \mathfrak{c}_k ,

$$\begin{aligned}\mathcal{M}_{\mathfrak{c}_m} &= \{\mathbb{C}/\mathbb{Z} + (\frac{1}{2} + \mathbf{i}b)\mathbb{Z} | b \in \mathbb{R}^{>0}\}, & \mathfrak{c}_m(z) &= \bar{z}, \\ \mathcal{M}_{\mathfrak{c}_k} &= \{\mathbb{C}/\mathbb{Z} + \mathbf{i}b\mathbb{Z} | b \in \mathbb{R}^{>0}\}, & \mathfrak{c}_k(z) &= \frac{1}{2} + \bar{z}.\end{aligned}$$

Consider \mathbb{P}^1 with the coordinate chart $z \in \mathbb{C} \cup \{\infty\}$ and let $w = \frac{1}{z}$. Let $\mathbb{P}_{\text{top}}^1$ and $\mathbb{P}_{\text{bot}}^1$ be two copies of \mathbb{P}^1 with coordinate charts z_t and z_b , respectively. Let

$$\Sigma_{\text{nod}} = \mathbb{P}_{\text{top}}^1 \cup \mathbb{P}_{\text{bot}}^1 / 0_t \sim 0_b, \infty_t \sim \infty_b$$

be a 2-nodal curve obtained by identifying 0 and ∞ of the two components. We equip Σ_{nod} with a real structure,

$$\sigma_{\text{nod}}: z_t \rightarrow \bar{z}_b,$$

which has the image of 0 and ∞ in Σ_{nod} , say q_0 and q_∞ , as the only fixed points. For $\epsilon_0, \epsilon_\infty \in \mathbb{R}$, let $\Sigma_{\epsilon_0, \epsilon_\infty}$ be the genus one Riemann surface obtained by smoothing the two nodes of Σ_{nod} via the gluing map

$$z_t z_b = \epsilon_0, \quad w_t w_b = \epsilon_\infty.$$

The involution σ_{nod} naturally extends to an involution $\sigma_{\epsilon_0, \epsilon_\infty}$ on $\Sigma_{\epsilon_0, \epsilon_\infty}$. Define

$$(2.1) \quad C_0 = \{|z_u| = \sqrt{\epsilon_0} \subset \Sigma_{\epsilon_0, \epsilon_\infty}\}, \quad C_\infty = \{|w_u| = \sqrt{\epsilon_\infty} \subset \Sigma_{\epsilon_0, \epsilon_\infty}\};$$

these two curves are invariant under $\sigma_{\epsilon_0, \epsilon_\infty}$ and are in $\text{Fix}(\sigma_{\epsilon_0, \epsilon_\infty})$ if and only if the corresponding gluing parameter is positive. Orient C_0 by $z_t = \sqrt{|\epsilon_0|}e^{-\mathbf{i}\mathbb{R}}$ and C_∞ by $z_b = \sqrt{|\epsilon_\infty|}e^{-\mathbf{i}\mathbb{R}}$.

Lemma 2.2. *For every $\epsilon_0, \epsilon_\infty \in \mathbb{R}$ with $0 < |\epsilon = \epsilon_0 \epsilon_\infty| < 1$, there is a biholomorphism $h_{\epsilon_0, \epsilon_\infty}$ from $(\Sigma_{\epsilon_0, \epsilon_\infty}, \sigma_{\epsilon_0, \epsilon_\infty})$ to $(\mathbb{C}/\mathbb{Z} + \zeta_\epsilon \mathbb{Z}, \sigma_\epsilon(z))$ where*

$$\zeta_\epsilon = \frac{-\mathbf{i} \ln(\epsilon)}{2\pi} \in \mathbb{H}, \quad \sigma_\epsilon(z) = \begin{cases} \bar{z} + \frac{1}{2}, & \text{if } \epsilon_1, \epsilon_2 < 0; \\ \bar{z}, & \text{otherwise,} \end{cases}$$

and such that $h_{\epsilon_0, \epsilon_\infty}(C_0) = \mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z} + \zeta_\epsilon \mathbb{Z}$ as oriented curves. Moreover, every genus one real curve is biholomorphic to some $(\Sigma_{\epsilon_0, \epsilon_\infty}, \sigma_{\epsilon_0, \epsilon_\infty})$, with a unique ϵ ; i.e. for each choice of parity of ϵ_1, ϵ_2 , $-\ln(|\epsilon|) \in \mathbb{R}^{>0}$ parameterizes the space of real genus one curves of the corresponding σ -type.

Proof. For every $\epsilon_0, \epsilon_\infty \in \mathbb{R}$, $\omega_\epsilon = \frac{idz_t}{z_t}$ extends to a holomorphic one-form on $\Sigma_{\epsilon_0, \epsilon_\infty}$. Let γ be the oriented simple closed curve given by

$$\begin{aligned} & \{z_t \in [\sqrt{|\epsilon_0|}, \frac{1}{\sqrt{|\epsilon_\infty|}}]\} \cup \{\text{sign}(\epsilon_0) \cdot w_b \in [\sqrt{|\epsilon_\infty|}, \frac{1}{\sqrt{|\epsilon_0|}}]\}, & \text{if } \epsilon > 0, \\ & \{z_t \in [\sqrt{\epsilon_0}, \frac{1}{\sqrt{\epsilon_\infty}}]\} \cup \{w_t = \sqrt{|\epsilon_\infty|}e^{i[0, \pi]}\} \cup \{w_b \in [\sqrt{|\epsilon_\infty|}, \frac{1}{\sqrt{\epsilon_0}}]\}, & \text{if } \epsilon_0 > 0, \epsilon_\infty < 0 \\ & \{z_t = e^{i[\pi, 2\pi]}\} \cup \{z_t \in [\sqrt{|\epsilon_0|}, \frac{1}{\sqrt{\epsilon_\infty}}]\} \cup \{w_b \in [\sqrt{\epsilon_\infty}, \frac{1}{\sqrt{|\epsilon_0|}}]\}, & \text{if } \epsilon_0 > 0, \epsilon_\infty < 0. \end{aligned}$$

Calculating the ratio of the periods of ω_ϵ along C_0 and γ , we get the stated result for ζ_ϵ . Given a genus 1 real curve (Σ, σ) , by lifting the action of σ to its universal cover \mathbb{C} , it is easy to show that (Σ, σ) is biholomorphic to $\mathbb{C}/\mathbb{Z} + \zeta_\epsilon \mathbb{Z}$, for a unique choice of ϵ . \square

We call a presentation $(\Sigma_{\epsilon_0, \epsilon_\infty}, \sigma_{\epsilon_0, \epsilon_\infty})$ of a genus one real curve, a **standard** presentation. It is easier to express the orientation argument in terms of standard presentations of a real tori. Next, we determine the real automorphism group $G_{\sigma, j}$, for every σ and j as above.

Lemma 2.3. *For each $\sigma = \mathbf{c}_a, \mathbf{c}_m, \mathbf{c}_k$ as above, $G_{\sigma, j}$ only depends on σ and*

$$G_\sigma \cong \begin{cases} \langle S^1, \text{flip}, \text{sw} \rangle, & \text{if } \sigma = \mathbf{c}_a \text{ or } \mathbf{c}_k; \\ \langle S^1, \text{flip} \rangle, & \text{if } \sigma = \mathbf{c}_m, \end{cases}$$

where

$$\text{flip}(z_t) = z_b, \quad \text{sw}(z_t) = \sqrt{\frac{\epsilon_\infty}{\epsilon_0}} w_t, \quad \text{sw}(z_b) = \sqrt{\frac{\epsilon_0}{\epsilon_\infty}} w_b,$$

and the S^1 part acts by

$$z_t \rightarrow e^{i\theta} z_t, \quad z_b \rightarrow e^{-i\theta} z_b.$$

Remark 2.4. In the affine presentation $T \cong \mathbb{C}/\Lambda$ of the real tori where $\Lambda \cong \mathbb{Z} + ib\mathbb{Z}$ if $\sigma = \mathbf{c}_a, \mathbf{c}_k$, and $\Lambda = \mathbb{Z} + (\frac{1}{2} + ib)\mathbb{Z}$ if $\sigma = \mathbf{c}_m$, flip and sw are given by

$$\text{flip}(z) = -z, \quad \text{sw}(z) = z + \frac{ib}{2};$$

and the S^1 part acts by the set of real translations

$$T_a: z \rightarrow z + a, \quad a \in \mathbb{R}/\mathbb{Z}.$$

Remark 2.5. Since $G_{\sigma, j}$ is independent of j , the extension of any $h \in G_{\sigma, j}$ to \mathcal{J}_σ is trivial. The action of automorphism group in higher genus is much more complicated as some $h \in G_{\sigma, j}$ might extend to a non-orientable orbifold structure near $j \in \mathcal{M}_\sigma$. In such a case, the parameter space will not be orientable.

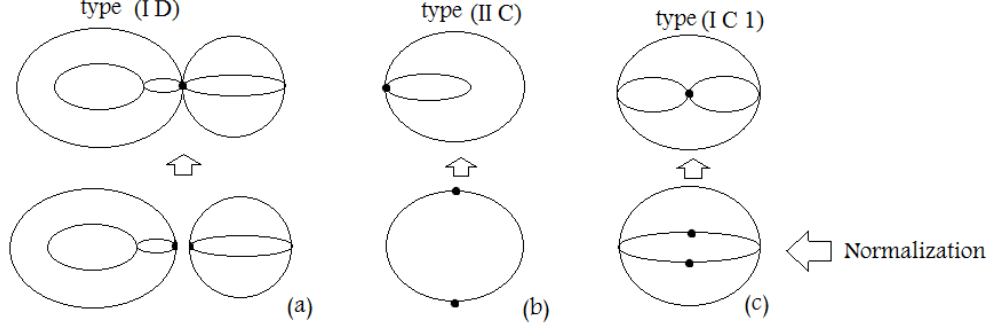


FIGURE 1. Possible types of 1-nodal genus one real curves.

3. COMPACTIFICATION

For each σ , let $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma}$ be the stable compactification of $\mathcal{M}_{g,k,\ell}(X, A)^{\phi, \sigma}$, it includes J -holomorphic real maps with nodal domain. The virtual codimension of a boundary stratum of $\overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma}$ is the number of nodes in the domains of the elements of that stratum; for

$$f = [u, \Sigma_{\text{nod}}, \sigma_{\text{nod}}, j_{\text{nod}}, (w_i)_{i=1}^k, (z_i, \sigma_{\text{nod}}(z_i))_{i=1}^\ell] \in \partial \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma},$$

if Σ_{nod} has s -nodes then $f \in \partial^s \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma}$. Therefore, elements of $\partial^1 \overline{\mathcal{M}}_{g,k,\ell}(X, A)^{\phi, \sigma}$ correspond to maps defined over 1-nodal real domains. In this case, let $q \in \Sigma_{\text{nod}}$ be the unique node of the domain. If there is only one node, it should be a fixed point of the real structure σ_{nod} on Σ_{nod} . Locally near q , Σ_{nod} is of the form

$$zw = 0 \subset \mathbb{C}^2.$$

We say q is of type (I), if σ_{nod} maps each component of $zw = 0$ to itself, and is of type (II), if it swaps the two components. In each case, after a holomorphic reparametrization, σ_{nod} near q has the form

$$(I) : z \rightarrow \bar{z}, \quad w \rightarrow \bar{w}, \quad (II) : z \rightarrow \bar{w}.$$

We call a node **disconnecting**, type D, if $\Sigma_{\text{nod}} \setminus \{q\}$ has two connected components and connecting, type C, otherwise. Let $(\tilde{\Sigma}_{\text{nod}}, \tilde{\sigma}_{\text{nod}})$ be the normalization of $(\Sigma_{\text{nod}}, \sigma_{\text{nod}})$. For a node of type (IC), we say it is of type (1) if

$$\{z \in \mathbb{R}\} \cup \{w \in \mathbb{R}\} \subset \text{Fix}(\tilde{\sigma}_{\text{nod}})$$

lies in one connected component of $\text{Fix}(\tilde{\sigma}_{\text{nod}})$, and is of type (2), otherwise. Figure 1 demonstrates possible types of 1-nodal genus one real curves. The nodal maps depicted in Figure 1(a), are elements of

$$(3.1) \quad \mathcal{M}_{1,k_1+1,\ell_1}(X, A_1)^{\phi, \sigma} \times_{(\text{ev}_1^{\mathbb{R}}, \text{ev}_1^{\mathbb{R}})} \mathcal{M}_{0,k_2+1,\ell_2}(X, A_2)^{\phi, \tau},$$

where σ can be either \mathbf{c}_a or \mathbf{c}_m . Those in Figure 1(b), are elements of

$$(3.2) \quad \mathcal{M}_{0,k,\ell+1}(X, A)^{\phi, \sigma} \times_{\text{ev}_1} L,$$

where σ can be either τ or η . Finally, those of Figure 1(c), are elements of

$$(3.3) \quad \mathcal{M}_{0,k+2,\ell}(X, A)^{\phi, \tau} \times_{(\text{ev}_1^{\mathbb{R}} \times \text{ev}_2^{\mathbb{R}}, \iota)} \Delta,$$

where $\iota: \Delta \hookrightarrow L \times L$ is the diagonal. This can happen if the starting σ is either \mathbf{c}_a or \mathbf{c}_m . In each case, for small $\epsilon \in \mathbb{R}$, let Σ_ϵ be the genus one Riemann surface obtained by smoothing the node q of Σ_{nod} via the gluing map $zw = \epsilon$. The involution σ_{nod} naturally extends to an involution σ_ϵ on Σ_ϵ . For the nodes of type (II) and (IC1), positive and negative values of ϵ result in different involutions. In fact,

$$\text{ind}(\sigma_{\epsilon > 0}) - \text{ind}(\sigma_{\epsilon < 0}) = 1.$$

For the nodes of type (ID), assuming $k = 0$, smoothing positively and negatively produce the same type of involution on the domain; but smoothing out the image curve in the positive and negative directions correspond to different curves in each direction; therefore, (3.1) is indeed a hypersurface. Let

$$f = f_1 \# f_2 \in \mathcal{M}_{1,1,\ell_1}(X, A_1)^{\phi, \sigma} \times_{(\text{ev}_1^{\mathbb{R}}, \text{ev}_1^{\mathbb{R}})} \mathcal{M}_{0,1,\ell_2}(X, A_2)^{\phi, \tau}.$$

Because of the stability condition, either $\ell_2 \neq 0$ or A_2 is non-trivial. If $\ell_2 \neq 0$, we fix one of the marked points; if u_2 is non-trivial and somewhere injective, we fix a somewhere injective point of the corresponding domain. By tracking the image of the chosen point, we see that gluing the map in positive and negative directions produce different J -holomorphic curves. If u_2 is multiple cover and $\ell_2 = 0$, then the obstruction bundle near $u_2 \in \mathcal{M}_{0,1,0}(X, A_2)^{\phi, \tau}$ is non-trivial and a Kuranishi neighborhood depends on the choice of \mathcal{E}_{u_2} of the following construction. By choosing \mathcal{E}_{u_2} properly, we can assure that gluing in different directions produce different maps. Therefore, the real codimension one boundary term (3.1), corresponding to $\epsilon = 0$, is indeed a hypersurface.

At last, in order to define invariants from this moduli space, we need to construct a virtual fundamental class. We can achieve this by putting a Kuranishi structure on the moduli space. Such a construction for $\overline{\mathcal{M}}_{0,k,\ell}(X, A)^{\phi, \tau}$ is described in [16, Section 7] and [4]; we only describe the necessary adjustments. For simplicity, we ignore the marked points until the end of this construction.

For $f = (u, \Sigma, \sigma, j) \in \mathcal{M}_{1,0,0}(X, A)^{\phi, \sigma}$, let

$$E_u \equiv u^*TX \rightarrow \Sigma, \quad E_u^{0,1} \equiv (T^*\Sigma)^{0,1} \otimes_{\mathbb{C}} E_u.$$

There are commutative diagrams

$$(3.4) \quad \begin{array}{ccc} E_u & \xrightarrow{T\phi} & E_u \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array} \qquad \begin{array}{ccc} E_u^{0,1} & \xrightarrow{T_\phi^1} & E_u^{0,1} \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\text{id}} & \Sigma \end{array}$$

where $T_\phi(v) = d\phi(v)$ and $T_\phi^1(\alpha) = d\phi \circ \alpha \circ d\sigma$. For a fixed complex structure j on Σ , the deformation theory of $\mathcal{M}_{1,0,0}(X, A)^{\phi, \eta}$ is described by the linearization of the Cauchy-Riemann operator,

$$(3.5) \quad L_{J,u}: W^{s,p}(E_u) \rightarrow W^{s-1,p}(E_u^{0,1}), \quad p > 2, s \geq 1;$$

see [11, Chapter 3] for a similar situation. If ∇ is the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$, $L_{J,u}$ can be written as

$$L_{J,u}(\xi) = \frac{1}{2}(\nabla\xi + J\nabla\xi \circ j) - \frac{1}{2}J(\nabla_\xi J)\partial_J(u).$$

There is a commutative diagram

$$\begin{array}{ccc} W^{s,p}(E_u) & \xrightarrow{L_{J,u}} & W^{s-1,p}(E_u^{0,1}) \\ \tilde{T}_\phi \downarrow & & \tilde{T}_\phi^1 \downarrow \\ W^{s,p}(E_u) & \xrightarrow{L_{J,u}} & W^{s-1,p}(E_u^{0,1}) \end{array}$$

where $\{\tilde{T}_\phi\xi\}(z) = T_\phi(\xi(\sigma(z)))$ and $\{\tilde{T}_\phi^1\alpha\}(z) = T_\phi^1(\alpha(z))$. Let

$$\begin{aligned} W^{s,p}(E_u)_\mathbb{R} &= \{\xi \in W^{s,p}(E_u) \mid \tilde{T}_\phi(\xi) = \xi\}, \\ W^{s-1,p}(E_u^{0,1})_\mathbb{R} &= \{\alpha \in W^{s-1,p}(E_u^{0,1}) \mid \tilde{T}_\phi^1(\alpha) = \alpha\} \end{aligned}$$

denote the spaces of **real** sections. Let $H^0(E_u)_\mathbb{R}$ and $H^1(E_u)_\mathbb{R}$ be the kernel and cokernel, respectively, of the restricted operator

$$L_{J,u}: W^{k,p}(E_u)_\mathbb{R} \rightarrow W^{k-1,p}(E_u^{0,1})_\mathbb{R}.$$

If $H^1(E_u)_\mathbb{R} = 0$, then $\mathcal{M}_{1,0,0}(X, A)^{\phi, \sigma}|_{\text{fixed } j}$ is a manifold near u of real dimension

$$\dim_\mathbb{R} H^0(E_u)_\mathbb{R} - \dim G_\sigma = \mathbb{R}(L_{J,u}) - 1 = c_1(A) - 1;$$

see [11, Theorem C.1.10]. Each pair of conjugate marked points increases the dimension by two and each real marked point increases the dimension by one, letting j to vary within the real one dimensional parameter space \mathcal{M}_σ we get the dimension formula (1.1).

If $H^1(E_u)_\mathbb{R} \neq 0$, we construct a Kuranishi chart around f . For this aim, we choose finite-dimensional complex subspaces

$$\mathcal{E}_u \subset W^{s,p-1}(E_u^{0,1})$$

such that

- (1) every $\xi \in \mathcal{E}_u$ is smooth and supported away from the boundary and marked points;
- (2) $\tilde{T}_\phi^1(\mathcal{E}_u) = \mathcal{E}_u$;
- (3) $L_{J,u}$ modulo \mathcal{E}_u is surjective.

We then choose our Kuranishi neighborhood to be $V(u) = [\bar{\partial}^{-1}(\mathcal{E}_u)]_{\mathbb{R}}$ (modulo G_σ , i.e. by stabilizing domain first), which is a smooth manifold of dimension

$$c_1(A) + k + 2\ell + \dim_{\mathbb{C}}(\mathcal{E}_u).$$

The obstruction bundle $\mathcal{E}(u)$ at each $f \in V(u)$ is obtained by parallel translation of \mathcal{E}_u with respect to the induced metric of J . We thus get a Kuranishi neighborhood $(V(u), \mathcal{E}(u))$ (together with a group action if the isotropy group is non-trivial). The Kuranishi map in this case is just the Cauchy-Riemann operator $f \rightarrow \bar{\partial}(f)$.

In order to construct Kuranishi charts for u in the boundary strata of $\overline{\mathcal{M}}_{1,k,\ell}(X, A)^{\phi, \sigma}$, we need gluing theorems as in [5, Chapter 7]. The gluing theorems are identical to those for J -holomorphic disks and spheres; we thus omit the details and refer the reader to [5].

This establishes the first part of Conjecture 1.3, i.e. that $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi}$ has the structure of a closed Kuranishi space. This part was in fact the well-known classical (and slightly controversial) part, the main goal of this note is to discuss the orientation problem.

4. LOW DIMENSIONAL EXAMPLES

Let X be a point, $k = 0$, $\ell = 1$ and $\sigma = \mathbf{c}_k$. Then, $\mathcal{M}_{1,0,1}(\text{pt}, \text{pt})^{\text{trivial}, \mathbf{c}_k}$ is nothing but $\mathcal{M}_{\mathbf{c}_k, 0, 1}$. Every $[\Sigma, (p, \mathbf{c}_k(p))]$ in $\mathcal{M}_{\mathbf{c}_k, 0, 1}$ is isomorphic to some $[\Sigma_b = \mathbb{C}/\mathbb{Z} + ib\mathbb{Z}, p]$, where (b, p) belongs to

$$\mathcal{P}_{\mathbf{c}_k, 0, 1} = \{(b, p) | b \in \mathbb{R}^{>0}, p \in i\mathbb{R}/ib\mathbb{Z}\}.$$

Via the map

$$(b, p) \rightarrow \zeta = be^{\frac{2\pi p}{b}},$$

we get an identification of $\mathcal{P}_{\mathbf{c}_k, 0, 1}$ and \mathbb{C}^* . There is an induced action of **flip** and **sw** on \mathbb{C}^* given by

$$\zeta \xrightarrow{\text{flip}} \bar{\zeta}, \quad \zeta \xrightarrow{\text{sw}} -\zeta,$$

such that

$$\mathcal{M}_{\mathbf{c}_k, 0, 1} \cong \mathbb{C}^* / \langle \text{flip}, \text{sw} \rangle.$$

Since the action of **flip** is orientation reversing, $\mathcal{M}_{\mathbf{c}_k, 0, 1}$ is not orientable. The situation for $\sigma = \mathbf{c}_a$ and \mathbf{c}_m is similar. That is why we need the dimension to be odd in Theorem 1.1; we need the complex dimension of X to be odd in order for the action of **flip** to be orientation preserving. We address this issue in more details in Section 5.

Next, we consider $X = \mathbb{P}^{2m-1}$, $k = \ell = 0$, $A = [2] \in H_2(\mathbb{P}^{2m-1}) \cong \mathbb{Z}$ and $\phi = \tau_{2m-1}$ or η_{2m-1} where

$$(4.1) \quad \tau_{2m-1}([z_1, z_2, \dots, z_{2m-1}, z_{2m}]) = ([\bar{z}_2, \bar{z}_1, \dots, \bar{z}_{2m}, \bar{z}_{2m-1}])$$

$$(4.2) \quad \eta_{2m-1}([z_1, z_2, \dots, z_{2m-1}, z_{2m}]) = ([-\bar{z}_2, \bar{z}_1, \dots, -\bar{z}_{2m}, \bar{z}_{2m-1}]).$$

In this case, generic $f = [u, \Sigma, \sigma] \in \mathcal{M}_{1,0,0}(\mathbb{P}^{2m-1}, [2])^\phi$ is a double cover of some real line

$$L \in \mathcal{M}_{0,0,0}(\mathbb{P}^{2m-1}, [1])^\phi \cong \text{Gr}_{\mathbb{R}}(2, 2m),$$

where $\text{Gr}_{\mathbb{R}}(2, 2m)$ is the Grassmannian of 2-dimensional real planes in \mathbb{R}^{2m} . Therefore, we may assume $m = 1$.

For each choice of ϕ on \mathbb{P}^1 , we can lift the action of ϕ to a complex conjugation involution (still denoted by ϕ)

$$\phi: \mathcal{O}_{\mathbb{P}^1}(2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(2),$$

and this induces an involution on $\mathcal{O}_{\mathbb{P}^1}(4)$. Let $H^0(\mathcal{O}(4))_{\mathbb{R}}$ be the set of real holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(4)$; $\dim_{\mathbb{R}} H^0(\mathcal{O}(4))_{\mathbb{R}} = 5$. For generic $s \neq 0 \in H^0(\mathcal{O}(4))_{\mathbb{R}}$, define

$$\Sigma_s = \{t \in \mathcal{O}_{\mathbb{P}^1}(2) \mid t^2 = s\}.$$

This is a ramified double cover of \mathbb{P}^1 , $\pi_s: \Sigma_s \rightarrow \mathbb{P}^1$, such that $g_{\Sigma_s} = 1$ and the involution ϕ on $\mathcal{O}_{\mathbb{P}^1}(2)$ induces a real structure σ_s on Σ_s . Therefore, $f_s = [\pi_s, \Sigma_s, \sigma_s] \in \mathcal{M}_{1,0,0}(\mathbb{P}^1, [2])^\phi$. For $s_1 \neq s_2 \in H^0(\mathcal{O}(4))_{\mathbb{R}}$, $f_{s_1} = f_{s_2}$ if and only if $s_2 = \lambda s_1$ for some positive constant λ . Note that if $\lambda \in \mathbb{R}^{<0}$, f_{s_2} and f_{s_1} are still biholomorphic to each other, but the biholomorphism does not commute with the induced real structures. Considering non-generic sections, it is easy to show that outside a real codimension-two subset

$$\mathcal{M}_{1,0,0}(\mathbb{P}^1, [2])^\phi \cong_{\text{outside codim } 2} (H^0(\mathcal{O}(4))_{\mathbb{R}} - \{0\})/\mathbb{R}^{>0} = S^4;$$

therefore, $\mathcal{M}_{1,0,0}(\mathbb{P}^1, [2])^\phi$ is orientable. Similarly, we can show that outside a real codimension-two subset, $\mathcal{M}_{1,0,0}(\mathbb{P}^{2m-1}, [2])^\phi$ is isomorphic to (an open subset of) some orientable S^4 -bundle over $\text{Gr}(2, 2m)$. Since non-real marked points come with complex orientation, we conclude that $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [2])^\phi$ is orientable. If m is even, this confirms the result of Theorem 1.1.

5. ORIENTATION, FOR EACH PIECE

In this section we prove Theorem 1.1. In the orientation problem for $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma}$, it is sufficient to consider the case $\ell = 0$, because any pair of marked points $(z_i, \sigma(z_i))$ increases the tangent space by $T_{z_i}\Sigma$, which has a canonical orientation.

Let $\mathcal{P}_{1,0,0}(X, A)^{\phi,\sigma}$ be the moduli space of parametrized J -holomorphic maps so that

$$\mathcal{M}_{1,0,0}(X, A)^{\phi,\sigma} \cong \mathcal{P}_{1,0,0}(X, A)^{\phi,\sigma}/G_\sigma.$$

For each $[f] = [u, \Sigma, \sigma, j] \in \mathcal{M}_{1,0,0}(X, A)^{\phi,\sigma}$, in order to put an orientation on

$$T_{[f]}\mathcal{M}_{1,0,0}(X, A)^{\phi,\sigma},$$

we should put an orientation on $T_f\mathcal{P}_{1,0,0}(X, A)^{\phi,\sigma}$ and $T_{\text{id}}G_\sigma$, whose quotient is invariant under the action of G_σ . Note that unlike the genus zero case, there is no canonical orientation on $T_{\text{id}}G_\sigma$ in this case. Let $\mathcal{P}_{1,0,0}(X, A, j)^{\phi,\sigma}$ be the moduli space of parametrized

J -holomorphic maps (u, Σ, σ, j) corresponding to a fixed complex structure j on the domain. Then

$$T_f \mathcal{P}_{1,0,0}(X, A)^{\phi, \sigma} = T_f \mathcal{P}_{1,0,0}(X, A, j)^{\phi, \sigma} \oplus T_j \mathcal{M}_\sigma.$$

For the orientation problem on $T_f \mathcal{P}_{1,0,0}(X, A, j)^{\phi, \sigma}$, we need to consider the determinant of the index bundle

$$\text{ind}_{\mathbb{R}} E = \Lambda^{\text{top}} H^0(E)_{\mathbb{R}} \otimes \Lambda^{\text{top}} (H^1(E)_{\mathbb{R}})^*,$$

where $E = u^* TX$, and $H^0(E)_{\mathbb{R}}$ and $H^1(E)_{\mathbb{R}}$ are the kernel and cokernel of a real Cauchy-Riemann operator on E . Recall (from the left diagram in (3.4)) that E admits an anticomplex linear involution T_ϕ .

Let (u, Σ, σ, j) be as above. By Lemma 2.2, we can assume $(\Sigma, \sigma, j) \cong (\Sigma_{\epsilon_0, \epsilon_\infty}, \sigma_{\epsilon_0, \epsilon_\infty}, j_\epsilon)$, for some $\epsilon_0, \epsilon_\infty$; this presentation is unique up to G_σ and the factorization $\epsilon = \epsilon_0 \epsilon_\infty$ of the unique ϵ determining the complex structure j_ϵ . The map

$$\mathcal{M}_\sigma \rightarrow \mathbb{R}, \quad \Sigma_{\epsilon_0, \epsilon_\infty} \rightarrow -\ln(|\epsilon|) \in \mathbb{R},$$

determines a parametrization and therefore an orientation on $T\mathcal{M}_\sigma$. Fix one such presentation; once it is fixed, it induces an orientation on $T_{\text{id}} G_\sigma$ via the isomorphism

$$T_{\text{id}} G_\sigma \cong \mathbb{R} \cdot v, \quad v = \frac{d}{d\theta} (e^{i\theta} z_t)|_{\theta=0}.$$

Let C_0 and C_∞ be the invariant circles as in the proof of Lemma 2.2. For δ very close to and smaller than 1, let A_0 and A_∞ be two small annuli around each of them given by

$$A_0 = \{\sqrt{\delta|\epsilon_0|} < |z_t| < \sqrt{|\epsilon_0|/\delta}\}, \quad A_\infty = \{\sqrt{\delta|\epsilon_\infty|} < |w_t| < \sqrt{|\epsilon_\infty|/\delta}\}.$$

Contracting the four circles $\partial A_0 \cup \partial A_\infty$, we obtain a nodal curve

$$\Sigma_{\text{nod}} = \Sigma_0 \cup \Sigma_\infty \cup \Sigma_{\text{top}} \cup \Sigma_{\text{bot}},$$

with an induced involution σ_{nod} ; see Figure 2.

Restriction of σ to C_0 and C_∞ is either identity or the antipodal map. In each case, A_* is invariant under the σ .

Definition 5.1. For $A = A_*$ as above, let $E \rightarrow A$ be a complex vector bundle with a real structure ϕ covering $\sigma|_A$. We call a trivialization of E over A ,

$$\begin{array}{ccc} E & \xrightarrow{\psi} & A \times \mathbb{C}^m \\ \pi \downarrow & & \downarrow \pi \\ A & \xrightarrow{\text{id}} & A \end{array}$$

admissible if the involution $\phi_\psi(z) = \psi_{\sigma(z)} \circ \phi \circ \psi_z^{-1}$ coincides with the standard involution $C: (z, v) \rightarrow (\sigma(z), \bar{v})$. Admissible trivializations ψ and ψ' of (E, ϕ) over A are called **homotopic** if there is a family of such trivializations ψ_t , $t \in [0, 1]$, such that $\psi_0 = \psi$ and $\psi_1 = \psi'$.

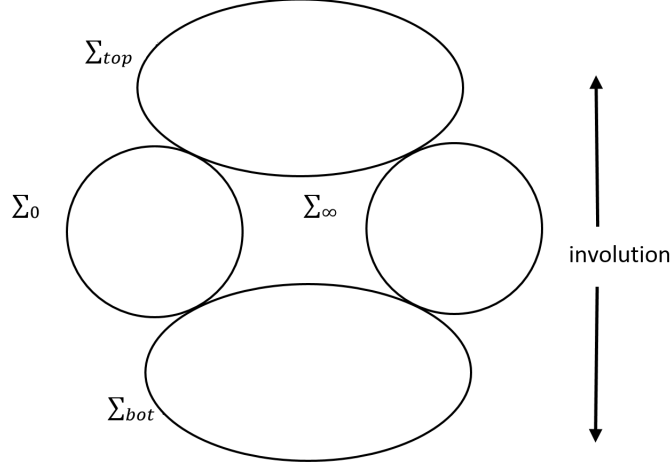


FIGURE 2. Degenerated elliptic curve.

The choice of a spin structure on TL or a real square root for K_X determines a unique homotopy class of admissible trivialization of E over A_* ; see [2, Section 2.1] and the proof of Theorem 1.3 there. Via the given trivialization, the bundle $(E, d\phi)$ descends to a bundle $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ over Σ_{nod} so that

$$E_{\text{nod}}|_{\Sigma_0 \text{ or } \Sigma_{\infty}} \cong \mathbb{P}^1 \times \mathbb{C}^n,$$

and

$$(d\phi)_{\text{nod}}|_{\Sigma_0 \text{ or } \Sigma_{\infty}} : (z, v) \rightarrow (\sigma_{\text{nod}}(z), \bar{v}).$$

Over $\Sigma_{\text{top}} \cup \Sigma_{\text{bot}}$, $(d\phi)_{\text{nod}}$ is an anticomplex linear map of the form

$$(d\phi)_{\text{nod}} : E_{\text{nod}}|_{\Sigma_{\text{top}}} \rightarrow E_{\text{nod}}|_{\Sigma_{\text{bot}}}.$$

A section of $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ is of the form $\xi = (\xi_0, \xi_{\infty}, \xi_{\text{top}}, \xi_{\text{bot}})$, with matching conditions at the nodes. Such a section is real if and only if

$$\xi_{\text{bot}}(\sigma_{\text{nod}}(z)) = (d\phi)_{\text{nod}}(\xi_{\text{top}}(z)) \quad \forall z \in \Sigma_{\text{top}} \text{ and } \xi_0 \in \Gamma(E_{\text{nod}}|_{\Sigma_0})_{\mathbb{R}}, \quad \xi_{\infty} \in \Gamma(E_{\text{nod}}|_{\Sigma_{\infty}})_{\mathbb{R}}.$$

Therefore, it is determined by an arbitrary section of $E_{\text{nod}}|_{\Sigma_{\text{top}}}$ and real sections of $E_{\text{nod}}|_{\Sigma_0}$ and $E_{\text{nod}}|_{\Sigma_{\infty}}$ which match at the nodes $q_{\text{top},0}$ and $q_{\text{top},\infty}$, the nodes between Σ_{top} and Σ_0 , Σ_{∞} , respectively. The matching condition at the nodes gives a short exact sequence

$$0 \rightarrow W^{1,p}(E_{\text{nod}})_{\mathbb{R}} \rightarrow W^{1,p}(E_{\text{nod}}|_{\Sigma_0} \oplus E_{\text{nod}}|_{\Sigma_{\infty}})_{\mathbb{R}} \oplus W^{1,p}(E_{\text{nod}}|_{\Sigma_{\text{top}}}) \rightarrow \mathbb{C}_{q_{\text{top},0}}^n \oplus \mathbb{C}_{q_{\text{top},\infty}}^n \rightarrow 0.$$

The associated index of the pair $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ is given by

(5.1)

$$\text{ind}_{\mathbb{R}} E_{\text{nod}} \cong \text{ind}_{\mathbb{R}}(E_{\text{nod}}|_{\Sigma_0}) \otimes \text{ind}_{\mathbb{R}}(E_{\text{nod}}|_{\Sigma_{\infty}}) \otimes \text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\Sigma_{\text{top}}}) \otimes \det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},0}}^n)^* \times \det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},\infty}}^n)^*.$$

Over Σ_0 and Σ_∞ , the index bundle is canonically isomorphic to (after deforming the Cauchy-Riemann operator) to

$$\Lambda^{\text{top}} H^0(\mathbb{P}^1 \times \mathbb{C}^n)_{\mathbb{R}} = \Lambda_{\mathbb{R}}^{\text{top}} \mathbb{R}^n \subset \Lambda_{\mathbb{C}}^{\text{top}} \mathbb{C}^n.$$

It inherits an orientation from the choice of trivialization. Since $\text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\Sigma_{\text{top}}})$ and

$$\det(\mathbb{C}_{q_{\text{top},0}}^n)^* \otimes \det(\mathbb{C}_{q_{\text{top},\infty}}^n)^*$$

carry orientations induced by their complex structures, they are canonically oriented. Thus, (5.1) induces an orientation on $\text{ind}_{\mathbb{R}} E_{\text{nod}}$. Then, a gluing argument analogous to that of [5, Proposition 8.1.4] determines an orientation on $\text{ind}_{\mathbb{R}} E$ itself.

Together with the provided orientation for $T_{\text{id}} G_\sigma$ and $T_j \mathcal{M}_\sigma$, this determines an orientation for $T_{[f]} \mathcal{M}_{1,0,0}(X, A)^{\phi,\sigma}$. Thus, it remains to show that the provided orientation is independent of the identification $(\Sigma, \sigma, j) \cong (\Sigma_{\epsilon_0, \epsilon_\infty}, \sigma_{\epsilon_0, \epsilon_\infty}, j_\epsilon)$. The orientation on $T_j \mathcal{M}_\sigma$ is obviously independent of the presentation because any $h \in G_\sigma$ extends to identity on \mathcal{M}_σ . Thus, we need to show that orientation constructed on the quotient

$$(5.2) \quad T_f \mathcal{P}_{1,0,0}(X, A, j)^{\phi,\sigma} / T_{\text{id}} G_\sigma$$

is independent of the chosen parametrization. For this, we need to show that the induced actions of \mathbf{flip} and \mathbf{sw} on (5.2) are orientation preserving.

Note that $\deg(E_{\text{nod}}|_{\Sigma_{\text{top}}}) = \frac{c_1(A)}{2}$; therefore by Riemann-Roch theorem

$$h^0(E_{\text{nod}}|_{\Sigma_{\text{top}}}) - h^1(E_{\text{nod}}|_{\Sigma_{\text{top}}}) = \frac{c_1(A)}{2} + \dim_{\mathbb{C}} X.$$

Action of \mathbf{flip} on Σ descends to Σ_{nod} ; we still denote it by \mathbf{flip} , such that $\mathbf{flip}: \Sigma_{\text{top}} \rightarrow \Sigma_{\text{bot}}$ sends z_{top} to z_{bot} and $\mathbf{flip}|_{\Sigma_0, \Sigma_\infty}$ sends $z \rightarrow \frac{1}{z}$. The induced action of \mathbf{flip} on $H^0(E_{\text{nod}}|_{\Sigma_0})_{\mathbb{R}}$ and $H^0(E_{\text{nod}}|_{\Sigma_\infty})_{\mathbb{R}}$ is identity.

Lemma 5.2. *The induced action of \mathbf{flip} on $\text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\Sigma_{\text{top}}})$ is equal to complex conjugation.*

Proof. By choosing \mathcal{E}_u big enough we may assume $H^1(E)_{\mathbb{R}} = 0$. Let

$$(\xi_{\text{top}}, \xi_{\text{bot}}) \in H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}} \oplus E_{\text{nod}}|_{\Sigma_{\text{bot}}})_{\mathbb{R}} \cong H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}}) \cong H^0(E_{\text{nod}}|_{\Sigma_{\text{bot}}}),$$

be a real section. Then $d\phi_{\text{nod}}(\xi_{\text{top}}(z_t)) = \xi_{\text{bot}}(\bar{z}_b)$. If we replace u by $u \circ \mathbf{flip}$, the corresponding action on (5.1) is given by replacing $H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}})$ by $H^0(E_{\text{nod}}|_{\Sigma_{\text{bot}}})$, i.e. replacing Σ_{top} with Σ_{bot} . Thus, the action of \mathbf{flip} on $H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}} \oplus E_{\text{nod}}|_{\Sigma_{\text{bot}}})_{\mathbb{R}}$ is given by

$$\begin{aligned} \mathbf{flip} : H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}}) &\rightarrow H^0(E_{\text{nod}}|_{\Sigma_{\text{bot}}}) \\ \mathbf{flip} : \xi_{\text{top}}(z_t) &\rightarrow \xi_{\text{bot}}(z_b) = d\phi_{\text{nod}}(\xi_{\text{top}}(\bar{z}_t)). \end{aligned}$$

Since $d\phi_{\text{nod}}$ is anticomplex linear, $\mathbf{flip}(a\xi_{\text{top}}(z_t)) = \bar{a}\mathbf{flip}(\xi_{\text{top}})$, for every $a \in \mathbb{C}$; therefore, $\mathbf{flip} : \Lambda^{\text{top}} H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}}) \rightarrow \Lambda^{\text{top}} H^0(E_{\text{nod}}|_{\Sigma_{\text{bot}}})$ is complex conjugation. \square

It is easy to see that the induced action of \mathbf{flip} on $\det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},0}}^n)^* \times \det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},\infty}}^n)^*$ is orientation preserving. Considering the effect of \mathbf{flip} on each individual term of (5.1), we obtain the following corollary.

Corollary 5.3. *If $\dim_{\mathbb{C}} X$ is odd and $\frac{c_1(A)}{2}$ is even, then the action of \mathbf{flip} on $T\mathcal{P}_{1,0,0}(X, A, j)^{\phi,\sigma}$ is orientation reversing.*

On the other hand, the action of \mathbf{flip} on $T_{\text{id}}G_{\sigma}$ is also orientation reversing; therefore, under the assumption of Theorem 1.1, the orientation on $T_{[f]}\mathcal{M}_{1,0,0}(X, A, j)^{\phi,\sigma}$ constructed above is independent of the action of \mathbf{flip} . This means changing the role of Σ_{top} and Σ_{bot} provides the same orientation on the tangent bundle of the moduli space.

Similar to \mathbf{flip} , action of \mathbf{sw} on Σ descends to Σ_{nod} ; we still denote it by \mathbf{sw} , such that $\mathbf{sw}: \Sigma_{\text{top}} \rightarrow \Sigma_{\text{top}}$ sends z_{top} to $\frac{1}{z_{\text{top}}}$, $\mathbf{sw}: \Sigma_{\text{bot}} \rightarrow \Sigma_{\text{bot}}$ sends z_{bot} to $\frac{1}{z_{\text{bot}}}$, and $\mathbf{sw}: \Sigma_0 \rightarrow \Sigma_{\infty}$ sends z_0 to z_{∞} . The induced action of \mathbf{sw} sends $H^0(E_{\text{nod}}|_{\Sigma_0})_{\mathbb{R}} \cong \mathbb{R}^{\dim_{\mathbb{C}} X}$ to $H^0(E_{\text{nod}}|_{\Sigma_{\infty}})_{\mathbb{R}} \cong \mathbb{R}^{\dim_{\mathbb{C}} X}$. Assuming $\dim_{\mathbb{C}} X$ is odd, we find that its effect on $H^0(E_{\text{nod}}|_{\Sigma_0})_{\mathbb{R}} \oplus H^0(E_{\text{nod}}|_{\Sigma_{\infty}})_{\mathbb{R}}$ is orientation reversing. Since the induced action of \mathbf{sw} on $H^0(E_{\text{nod}}|_{\Sigma_{\text{top}}}) \otimes H^1(E_{\text{nod}}|_{\Sigma_{\text{top}}})^*$ is complex linear, it is orientation preserving. Finally, it is easy to see that the action of \mathbf{sw} on $\det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},0}}^n)^* \times \det_{\mathbb{C}}(\mathbb{C}_{q_{\text{top},\infty}}^n)^*$ is complex linear, therefore, it is orientation preserving.

Corollary 5.4. *If $\dim_{\mathbb{C}} X$ is odd, then the action of \mathbf{sw} on $T\mathcal{P}_{1,0,0}(X, A, j)^{\phi,\sigma}$ is orientation reversing.*

As in the previous case, the action of \mathbf{sw} on $T_{\text{id}}G_{\sigma}$ is orientation reversing; therefore, under the assumption of Theorem 1.1, the orientation on $T_{[f]}\mathcal{M}_{1,0,0}(X, A, j)^{\phi,\sigma}$ constructed above is independent of the action of \mathbf{sw} .

Putting together, we conclude that under the assumptions of Theorem 1.1, $T\mathcal{M}_{1,0,0}(X, A)^{\phi,\sigma}$ and therefore $T\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma}$ is orientable.

6. GENUS ONE REAL INVARIANTS

In Section 3, we showed that $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi}$ has a Kuranishi structure without real codimension-one boundary and in Section 5, assuming that $\dim_{\mathbb{C}} X$ is odd, $4|c_1(TX)$, $\text{Fix}(\phi)$ is spin, and K_X has a real square root, we showed that each component $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma}$ of $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi}$ is orientable.

The next three lemmas (last one being a conjecture) state that these three orientations are compatible along the common boundaries given by (3.1), (3.2), (3.3). If all of them are true, we can then conclude that $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi}$ is orientable. The first lemma below implies that the orientation of $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma}$, $\sigma = \mathbf{c}_a, \mathbf{c}_m$, extends across the hypersurface (3.1). The second and third lemmas imply that the orientations of $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma_{\epsilon>0}}$ and $\mathcal{M}_{1,0,\ell}(X, A)^{\phi,\sigma_{\epsilon<0}}$ can be extended across the common boundaries (3.2) and (3.3), possibly after flipping the orientation of the later. We consider the induced orientation on the boundary ∂M of an oriented manifold M to be the one

given by the inward normal vector field; i.e. $TM|_{\partial M} = T\partial M \times v_{\text{in}}$, as oriented vector spaces.

Lemma 6.1. *Let (X, ω, ϕ) be a symplectic manifold with a real structure such that $L = \text{Fix}(\phi)$ is spin, K_X has real square root, and that $4|c_1(TX)$. Then the gluing maps*

$$\mathcal{M}_{1,1,\ell_1}(X, A_1)^{\phi, \sigma} \times_{(\text{ev}_1^{\mathbb{R}}, \text{ev}_1^{\mathbb{R}})} \mathcal{M}_{0,1,\ell_2}(X, A_2)^{\phi, \tau} \times \begin{cases} \mathbb{R}^{\geq 0} \\ \mathbb{R}^{\leq 0} \end{cases} \rightarrow \overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \sigma}$$

given by smoothing domain with respect to the corresponding gluing parameter ϵ , are orientation-preserving.

Proof of this lemma is identical to that of [2, Lemma 3.1], which indeed corresponds to the fact that the action of $\tau_{\mathcal{M}}$ (see [2, Section 1.2] for the definition) on the double cover $\mathcal{M}_{1,\ell_2}^{\text{disk}}(X, A_2)_{\text{dec}}$ of $\mathcal{M}_{0,1,\ell_2}(X, A_2)^{\phi, \tau}$ is orientation reversing; see [6, Corollary 5.6].

Lemma 6.2. *Let (X, ω, ϕ) be a symplectic manifold with a real structure such that $L = \text{Fix}(\phi)$ is spin, $4|c_1(TX)$, K_X has real square root, and $\dim_{\mathbb{C}} X$ is odd. Then the gluing maps*

$$(\mathcal{M}_{1,0,\ell+1}(X, A)^{\phi, \sigma_{\text{nod}}} \times_{\text{ev}_1} L) \times \begin{cases} \mathbb{R}^{\geq 0} \\ \mathbb{R}^{\leq 0} \end{cases} \rightarrow \begin{cases} \overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \sigma_+} \\ \overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \sigma_-} \end{cases}$$

are orientation-preserving, provided the Lagrangian on the left-hand side is oriented by the chosen spin structure of TL in the first case and by the opposite of the choice of the isomorphism

$$\Lambda_{\mathbb{R}}^{\text{top}}(TL) \cong (\Lambda_{\mathbb{C}}^{\text{top}}(TL))_{\mathbb{R}} = (K_X^*)_{\mathbb{R}} \approx E_{\mathbb{R}}^* \otimes E_{\mathbb{R}}^*,$$

with $(K_X, \phi_{K_X}) \cong (E \otimes E, \phi_E \otimes \phi_E)$, in the second case.

Proof. For the proof, we may assume $\ell = 0$. A curve in the common boundary of these two moduli spaces is of the form

$$f = [u, \Sigma_{\text{nod}}, \sigma_{\text{nod}}],$$

where Σ_{nod} is equal to \mathbb{P}^1 with 0 and ∞ identified. Let $(\tilde{\Sigma}_{\text{nod}}, \tilde{\sigma}_{\text{nod}})$ be the normalization of $\Sigma_{\text{nod}}, \sigma_{\text{nod}}$. There are two options for σ_{nod} , either $\tilde{\sigma}_{\text{nod}} = \tau$ or $\tilde{\sigma}_{\text{nod}} = \eta$. In each case, the point $q = 0 \equiv \infty$ is a fix point of σ_{nod} . We replace each such f with the unstable map

$$f_{\text{un}} = [u_{\text{un}}, \Sigma_{\text{un}} = \tilde{\Sigma}_{\text{nod}} \cup \mathbb{P}_{\text{un}}^1, \sigma_{\text{un}}],$$

with u_{un} restricting to the constant $u(q)$ over the unstable part \mathbb{P}_{un}^1 . The nodal domain of f_{un} is a union of two \mathbb{P}^1 's with the two 0 and the two ∞ identified. We can view f_{un} as an element of $\partial \overline{\mathcal{M}}_{1,0,0}(X, A)^{\phi, \sigma_+}$ by extending the involution to \mathbb{P}_{un}^1 via $\sigma_{\text{un}}|_{\mathbb{P}_{\text{un}}^1} = \tau$ and as an element of $\partial \overline{\mathcal{M}}_{1,0,0}(X, A)^{\phi, \sigma_-}$ by extending the involution to \mathbb{P}_{un}^1 via $\sigma_{\text{un}}|_{\mathbb{P}_{\text{un}}^1} = \eta$. The connected component of the identity in the real automorphism group of f_{un} , restricted to the unstable component, is isomorphic S^1 . Following the orientation and gluing

argument in [5, Section 8.3] and [5, Section 7.4.1], it is easy to see that the index bundles are isomorphic and the isomorphism is orientation-preserving.

In fact, for each glued map f_ϵ over the glued domain $(\Sigma_\epsilon, \sigma_\epsilon, j_\epsilon)$, in order to orient the tangent space $T_{f_\epsilon} \mathcal{M}_{1,0,0}(X, A)^{\phi, \sigma_\epsilon}$, we fix an admissible trivialization of $f_\epsilon^* TX$ over a neighborhood of two invariant circles, C_0 and C_∞ in Σ_ϵ , respectively, degenerate Σ_ϵ into $\Sigma'_{\text{nod}} = \mathbb{P}_0^1 \cup \mathbb{P}_\infty^1 \cup \mathbb{P}_{\text{top}}^1 \cup \mathbb{P}_{\text{bot}}^1$ and degenerate $u_\epsilon^* TX$ into a bundle over Σ'_{nod} , such that over $\mathbb{P}_0^1 \cup \mathbb{P}_\infty^1$ the induced bundle is admissibly trivial. By definition, Σ_{un} is equal to Σ_ϵ degenerated along the boundaries of neighborhood A_0 of C_0 , and Σ'_{nod} is obtained from Σ_{un} by degenerating $\tilde{\Sigma}_{\text{nod}}$ along the boundaries of a neighborhood A_∞ of $C_\infty \subset \tilde{\Sigma}_{\text{nod}}$ into $\mathbb{P}_\infty^1 \cup \mathbb{P}_{\text{top}}^1 \cup \mathbb{P}_{\text{bot}}^1$; in this situation \mathbb{P}_0^1 would be \mathbb{P}_{un}^1 . Over \mathbb{P}_{un}^1 , $u_{\text{un}}|_{\mathbb{P}_{\text{un}}^1}$ is trivial and the set of real sections are orientably isomorphic to $T_{u(q)} L$. This oriented isomorphism determines the orientation of the index bundle of the glued map.

Therefore, in order to finish the comparison, it remains to compare the connected components of the identity in the automorphism groups of the domains before and after the gluing and the change in the complex structure of the domain. Let

$$G_0 = \text{Aut}_{\mathbb{R}}([\Sigma_{\text{nod}}])^0 \cong S^1$$

be the identity component of the real automorphism group of Σ_{nod} . This is a real 1-dimensional Lie group which inherits an orientation after we choose one of the pre-images of q in $\tilde{\Sigma}_{\text{un}}$ as $0 \in \tilde{\Sigma}_{\text{nod}} \cong P^1$. Then,

$$(6.1) \quad T_f(\mathcal{P}_{0,0,1}(X, A)^{\phi, \tilde{\sigma}_{\text{nod}}} \times_{\text{ev}_1} L) \cong T_f(\mathcal{M}_{0,0,1}(X, A)^{\phi, \tilde{\sigma}_{\text{nod}}} \times_{\text{ev}_1} L) \oplus T_{\text{id}} G_0$$

as oriented vector spaces, with the isomorphism obtained by splitting the short exact sequence

$$0 \rightarrow T_{\text{id}} G_0 \rightarrow T_f(\mathcal{P}_{0,0,1}(X, A)^{\phi, \tilde{\sigma}_{\text{nod}}} \times_{\text{ev}_1} L) \cong T_f(\mathcal{M}_{0,0,1}(X, A)^{\phi, \tilde{\sigma}_{\text{nod}}} \times_{\text{ev}_1} L) \rightarrow 0.$$

After replacing Σ with Σ_{un} , the real automorphism group of the domain increases by a factor of S^1 and the gluing parameter of the domain takes values in \mathbb{C} . To kill the extra S^1 -action in both the automorphism group and the gluing parameter, as in the statement of the lemma, we consider the gluing parameter to be positive real (absolute value of the complex one) and restrict to a real 1-dimensional section of $G_0 \times S^1$ given by $G'_0 = \{(e^{-i\theta}, e^{i\theta})\} \subset G_0 \times S^1$, which is canonically isomorphic to G_0 . Let

$$\mathcal{C} = \{(z_{\text{un}}, z, \epsilon) \in \mathbb{P}_{\text{un}}^1 \times \tilde{\Sigma}_{\text{nod}} \times \mathbb{R}^{\geq 0} | z_{\text{un}} z = \epsilon\}.$$

This is a real one-parameter family of genus one real curves over $\mathbb{R}^{\geq 0}$,

$$(z_{\text{un}}, z, \epsilon) \rightarrow \zeta_\epsilon = -\frac{i \ln(\epsilon)}{\pi},$$

that describes a gluing of the singular fiber Σ_{un} into smooth real curves. Action of G_0 extends to the whole family by

$$(z_{\text{un}}, z, \epsilon) \rightarrow (e^{-i\theta} z_{\text{un}}, e^{i\theta} z, \epsilon).$$

By convention of Section 2, the map

$$\mathcal{C} \rightarrow \mathbb{R}^{>0}, \quad (z_{\text{un}}, z, \epsilon) \rightarrow -\ln(\epsilon),$$

defines the orientation of the space of complex structures $\mathcal{M}_{\sigma_{\pm}}$. Therefore, $\frac{\partial}{\partial \epsilon}$ lifts to inward normal vector field along the boundary and

$$T_f(\mathcal{P}_{0,0,1}(X, A)^{\phi, \tilde{\sigma}_{\text{nod}}} \times_{\text{ev}_1} L) \oplus \frac{\partial}{\partial \epsilon} \cong T_f \mathcal{M}_{1,0,0}(X, A)^{\phi, \sigma_{\pm}}$$

is an isomorphism of oriented vector spaces. This establishes the claim. \square

Conjectural Lemma 6.3. *Let (X, ω, ϕ) be a symplectic manifold with a real structure such that $L = \text{Fix}(\phi)$ is spin, K_X has real square root, $4|c_1(TX)$, and $\dim_{\mathbb{C}} X$ is odd. Then the gluing maps*

$$(6.2) \quad (\mathcal{M}_{0,2,\ell}(X, A)^{\phi, \tau} \times_{\text{ev}_1^{\mathbb{R}} \times \text{ev}_2^{\mathbb{R}}, \ell} \Delta) \times \begin{cases} \mathbb{R}^{\geq 0} \\ \mathbb{R}^{\leq 0} \end{cases} \rightarrow \begin{cases} \overline{\mathcal{M}}_{1,0,l}(X, A)^{\phi, \epsilon_a} \\ \overline{\mathcal{M}}_{1,0,l}(X, A)^{\phi, \epsilon_m} \end{cases}$$

given by smoothing domain with respect to the corresponding gluing parameter ϵ , are orientation-preserving, provided the diagonal $\Delta \cong L$ in the fiber product is oriented by the chosen spin structure of TL in the first case and by the opposite of that in the second case.

Remark 6.4. Similar to before, we may assume $\ell = 0$. In order to orient the moduli space $\mathcal{M}_{0,2,0}(X, A)^{\phi, \tau}$, we need to orient $\text{Fix}(\tau) = S^1$ and put an ordering on the two real marked points. Since $k = 2$, changing the orientation of $\text{Fix}(\tau) = S^1$ does not change the resulting orientation of the moduli space; therefore, the orientation of $\mathcal{M}_{0,2,0}(X, A)^{\phi, \tau}$ is independent of the choice of the orientation of $\text{Fix}(\tau)$. Let G_2 be the automorphism group of disk (half of \mathbb{P}^1) with two boundary marked points. This is equal to connected component of identity in the real automorphism group of (\mathbb{P}^1, τ) with two real marked points. The group G_2 is a real 1-dimensional Lie group isomorphic to \mathbb{R}^* . The choice of isomorphism and therefore the choice of orientation on G_2 depends on the ordering of the marked points. Changing the ordering, changes the orientation of the parametrized moduli space; it also changes the orientation of $T_{\text{id}} G_2$. Therefore, the orientation of $\mathcal{M}_{0,2,0}(X, A)^{\phi, \tau}$ does not depend on the choice of ordering of the two real marked points. Therefore, we can choose an ordering on these marked points and also choose an orientation of $\text{Fix}(\tau)$ (which is equal to picking one of the disks in $\mathbb{P}^1 \setminus \text{Fix}(\tau)$).

Remark 6.5. Let

$$\pi: \mathcal{C} \rightarrow (-\epsilon, \epsilon), \quad \Sigma_{\epsilon} = \pi^{-1}(\epsilon),$$

be the one parameter family of real curves obtained by smoothing the nodal domain of f . Let $\sigma_{\mathcal{C}}$ be the involution on the total space of \mathcal{C} and $\sigma_{\epsilon} = \mathcal{C}|_{\Sigma_{\epsilon}}$. Then $P_{\mathbb{R}} = \text{Fix}(\sigma_{\mathcal{C}})$ is a pair of pants; i.e. for $\epsilon > 0$, $\text{Fix}(\sigma_{\epsilon}) \cong S^1 \cup S^1$, and for $\epsilon < 0$, $\text{Fix}(\sigma_{\epsilon}) \cong S^1$. Assume

$$u_{\epsilon}: (\Sigma_{\epsilon}, \sigma_{\epsilon}) \rightarrow (X, \phi),$$

is a real gluing of f into smooth curves in $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \mathfrak{c}_a}$ and $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \mathfrak{c}_m}$. By abuse of notation, we let $P_{\mathbb{R}}$ to also denote for its image inside L . The choice of spin structure on L gives a trivialization of $TL|_{P_{\mathbb{R}}}$.

Note that the orientation on (6.2) does not depend on the choice of the square root for K_X . This means, near this particular boundary of $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \mathfrak{c}_m}$, there should exist a canonical choice of square root.

Similarly, define \mathbb{P}_i to be the pair of pants submanifold of \mathcal{C} obtained by tracing the image of imaginary line in \mathcal{C} after gluing. For $\epsilon > 0$, $P_i \cap \Sigma_{\epsilon}$ is union of two disjoint circles, and for $\epsilon < 0$, $P_i \cap \Sigma_{\epsilon}$ is the non-fixed invariant circle of \mathfrak{c}_m . For $\epsilon > 0$, an admissible choice of trivialization of u^*TX over $P_i \cap \Sigma_{\epsilon}$ depends on an arbitrary choice of the trivialization on one of the S^1 components. Thus, there are \mathbb{Z} homotopy classes of such trivialization. For $\epsilon = 0$, an admissible choice of the trivialization of u^*TX over $P_i \cap \Sigma_0$ depends on an arbitrary choice of the trivialization on one of the S^1 components such that restricted to the node, it takes the conjugation to complex conjugation on \mathbb{C}^n . Every other homotopy class of an admissible trivialization is given by a loop of complex matrices which is real at the node. Thus, there are $\mathbb{Z} \oplus \mathbb{Z}_2$ homotopy classes of such trivializations, where the \mathbb{Z}_2 component is given by the sign of the determinant of the real matrix above. Those homotopy classes for which the induced trivialization of TL at the node coincides with the given orientation of TL form the distinguished \mathbb{Z} -component of $\mathbb{Z} \oplus \mathbb{Z}_2$. For $\epsilon < 0$, every admissible choice of the trivialization of u^*TX over $P_i \cap \Sigma_0$, via cobordism P_i , extends to an admissible choice of the trivialization of u^*TX over $P_i \cap \Sigma_{\epsilon}$. The corresponding map on the set of homotopy classes is the projection on the \mathbb{Z}_2 -component. Therefore, the image of the distinguished component determines a unique homotopy class of the admissible trivialization. We conclude that, near the boundary (6.2), there is a distinguished choice of the admissible trivialization of u^*TX over the non-fixed invariant curve of σ_{ϵ} , if $\epsilon < 0$, which only depends on the choice of the orientation of L . Note that the orientation on L , itself, is determined by the choice of the square root; see [2, Section 2.2]. Taking the limit of the admissible trivialization of $u_{\epsilon}^*TX|_{P_i \cap \Sigma_{\epsilon}}$, we find that the distinguished choice above, is indeed the one implied by the square root.

7. ODD DIMENSIONAL PROJECTIVE SPACE

In the rest of this paper, we use equivariant localization to calculate genus one real invariants of odd dimensional projective space, by summing over the fixed loci of a torus action on $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi}$. As in [12, Section 3], some of these loci, which we call **separable**, are described by graphs with two half-edges; while for some others (of \mathfrak{c}_k -type), which we call **non-separable**, there is no canonical way of dividing the real curve into halves. In the separable case, the contribution of the complement of the half-edges to the normal bundle of the corresponding fixed loci is standard. Most of the calculations are similar to the genus zero case, which we will recall from [2, Appendix]. There, we explicitly describe the canonical square root structure on $K_{\mathbb{P}^{4m-1}}$ and the spin

structure on \mathbb{RP}^{4m-1} and build an explicit orientation for moduli spaces over \mathbb{P}^{4m+1} . In the non-separable case, the orientating procedure of Section 5 does not naturally appear. In Appendix A, we develop a new technique of orienting the corresponding component of the moduli space, which is suitable for the calculations over the set of non-separable real curves of \mathfrak{c}_k -type. We describe the fixed loci of a natural action of

$$\mathbb{T} \equiv (S^1)^m \equiv \{(\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m : |\zeta_k| = 1\}$$

on $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^\phi$ in Section 7.1 and their normal bundles in Section 7.2. In Section 7.3, we then compute some low-degree genus one real invariants.

7.1. Fixed loci. For $i = 1, 2, \dots, 2m$, we define

$$\bar{i} = \begin{cases} i+1, & \text{if } 2 \nmid i; \\ i-1, & \text{if } 2 \mid i. \end{cases}$$

The m -torus \mathbb{T} acts on \mathbb{P}^{2m-1} by

$$(\zeta_1, \dots, \zeta_m) \cdot [z_1, z_2, \dots, z_{2m-1}, z_{2m}] = [\zeta_1 z_1, \zeta_1^{-1} z_2, \dots, \zeta_m z_{2m-1}, \zeta_m^{-1} z_{2m}].$$

This action commutes with the involutions $\phi = \tau_{2m-1}, \eta_{2m-1}$ and has $2m$ fixed points,

$$p_1 = [1, 0, \dots, 0], \quad \dots \quad p_{2m} = [0, \dots, 1].$$

We note that $\phi(p_i) = p_{\bar{i}}$. By composition on the left, \mathbb{T} also acts on $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi,c}$, where $c = \mathfrak{c}_a, \mathfrak{c}_m$, or \mathfrak{c}_k .

Lemma 7.1. ([13, Lemma 3.1]) *The irreducible and reduced \mathbb{T} -fixed curves in \mathbb{P}^{2m-1} are the lines L_{ij} connecting the points p_i and p_j with $i \neq j$. Moreover, the irreducible and reduced ϕ - and \mathbb{T} -fixed curves in \mathbb{P}^{2m-1} are the lines $L_{i\bar{i}}$.*

Let $\lambda_i \in H_{\mathbb{T}}^*$ be the equivariant first Chern class of $\mathcal{O}_{\mathbb{P}^{2m-1}}|_{p_i}$. Thus,

$$\lambda_{\bar{i}} = -\lambda_i, \quad H_{\mathbb{T}}^* = \mathbb{Q}[\lambda_1, \lambda_3, \dots, \lambda_{2m-1}].$$

Let $[f, \Sigma, (z_k, c(z_k))_k]$ be an element of $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi,c}$ fixed by the \mathbb{T} -action. Since there are no \mathbb{T} -fixed points in \mathbb{P}^{2m-1} that are also fixed by ϕ , if $c = \mathfrak{c}_a$ or \mathfrak{c}_m , the domain Σ of f contains two central components, which are fixed under the involution, whose union we denote by Σ_0 , while the remaining irreducible components come in conjugate pairs. If $c = \mathfrak{c}_k$, in addition to the above situation, which we call the **separable** case, there is a **non-separable** case in which the involution over Σ has no fixed component; in this case, we can still choose a conjugate pair of non-contracted components (the choice is not unique), whose union we denote by Σ_0 , then the remaining irreducible components come in conjugate pairs; or, we can choose two conjugate nodes, p_0 and \bar{p}_0 , removing which divides Σ into two connected components conjugate to each other. Unless some other choices are made, the choice of conjugate nodes (p_0, \bar{p}_0) is not unique. Furthermore, in the separable case, f_0 restricted to each component Σ_{0j} of Σ_0 is a cover of some line $L_{i\bar{i}}$ of some degree $d_{0j} \in \mathbb{Z}^+$ which is branched only over p_i and $p_{\bar{i}}$. In both cases, since $g(\Sigma) = 1$ and Σ is symmetric with respect to an involution, all components of Σ should

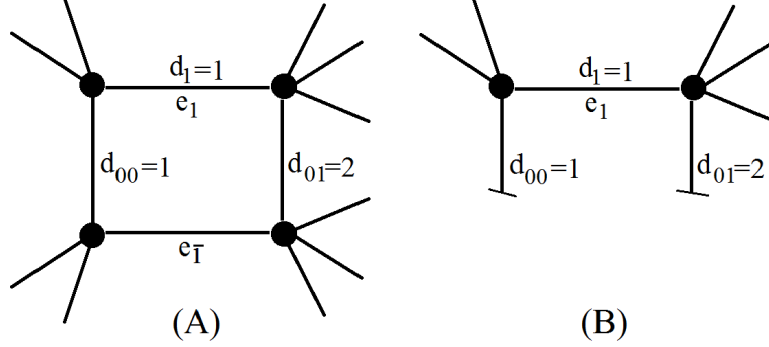


FIGURE 3. A separable genus 1 real curve and one of its halves.

be rational. Every nodal and marked point of Σ and branched point of f is mapped to a fixed point p_j .

If $d_0 = d_{01} + d_{02} < d$ or $\ell > 1$, the complement of Σ_0 in Σ consists of two genus zero nodal curves Σ' and Σ'' , each with $\ell+2$ marked points $(x_k)_{k=1}^{\ell+2}$ so that x_1 corresponds to the node shared with Σ_{01} , x_2 corresponds to the node shared with Σ_{02} and each of the remaining points is decorated by a sign s_k , $+$ or $-$, depending on whether it is the first or the second point in the pair $(z_{k-2}, c(z_{k-2}))$. Similarly, in the non-separable case, each half of Σ together with (p_0, \bar{p}_0) is a genus zero nodal curve with $\ell+2$ marked points $(x_k)_{k=1}^{\ell+2}$ so that $x_1 = p_0$, $x_{\ell+2} = \bar{p}_0$, and each of the remaining points is decorated by a sign s_k , $+$ or $-$, depending on whether it is the first or the second point in the pair $(z_{k-2}, c(z_{k-2}))$.

In the separable case, similar to [8, Section 27], every fixed locus of such maps can be modeled on a labeled genus one graph, Γ , symmetric about the mid-points of a distinguished edges e_{0j} which correspond to the central components Σ_{0j} of the \mathbb{T} -fixed maps in the fixed locus. Every edge e of Γ is labeled by some $d_e \in \mathbb{Z}^+$, indicating the degree of the corresponding map; these labels are preserved by the reflection symmetry of Γ . Every vertex v is labeled by some $j_v = 1, 2, \dots, 2m$ in such a way that the reflection symmetry takes a vertex labeled j to a vertex labeled \bar{j} . The graph Γ also contains open edges which correspond to the marked points of the domain Σ ; we denote by $v(k)$ the vertex to which the k -th marked point is attached. Figure 3(A) shows one such graph describing a separable $\text{Fix}(\mathbb{T})$ -locus in $\overline{\mathcal{M}}_{1,0,10}(\mathbb{P}^{2m-1}, [5])^{r_{2m-1}, c_a}$. Removing e_{0i} 's from Γ , we get a disconnected graph $\Gamma' \sqcup \bar{\Gamma}'$, where $\bar{\Gamma}'$ is the graph obtained from Γ' by replacing each vertex label j by \bar{j} . Choose one of the connected subgraphs, e.g. Γ' , and add the corresponding half-edges in place of the central edges; see Figure 3(B). We denote the total half graph by Γ_{half} . All calculations below are based on this half-graph; it is straightforward (also implied by Conjecture 1.3) to check that the result is independent of which half we choose.

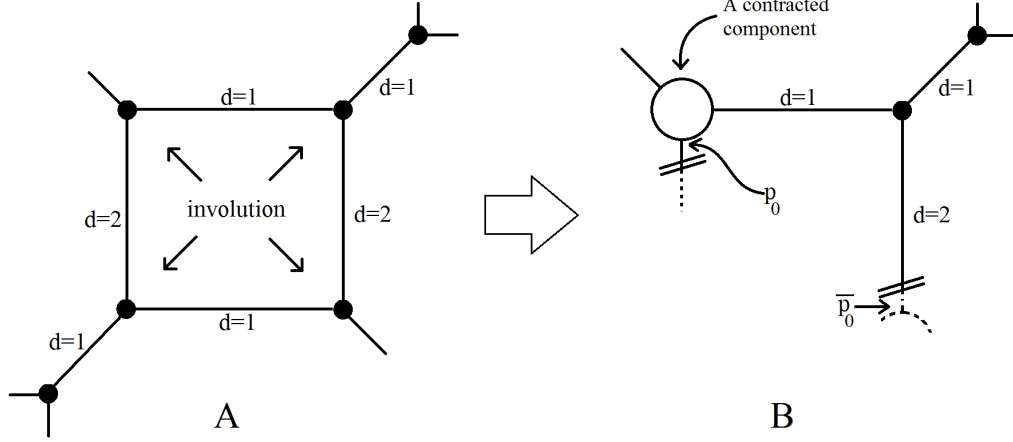


FIGURE 4. A non-separable genus 1 real curve and one possible half.

Similarly, after choosing a pair of conjugate nodes (p_0, \bar{p}_0) , we can divide a \mathfrak{c}_k -type non-separable Γ into two halves and use one for the calculations; by the results of Appendix A, the result of calculations is independent of the choice of conjugate nodes or the half. Figure 4(A) shows one such graph describing a non-separable $\text{Fix}(\mathbb{T})$ -locus in $\overline{\mathcal{M}}_{1,0,3}(\mathbb{P}^{2m-1}, [8])^{\tau_{2m-1}, \mathfrak{c}_k}$. Figure 4(B) shows one possible half of Figure 4(A).

For each vertex v in Γ_{half} , let $\overline{\mathcal{M}}_v = \overline{\mathcal{M}}_{0, \text{val}(v)}$, where $\text{val}(v)$ is the valence of v , i.e. the number of edges and open edges in Γ leaving v , and $\overline{\mathcal{M}}_{0, \text{val}(v)}$ denotes a point if $\text{val}(v) = 1, 2$. Let

$$\overline{\mathcal{M}}_{\Gamma_{\text{half}}} = \prod_v \overline{\mathcal{M}}_v, \quad D_{\Gamma_{\text{half}}} = d_0 \cdot \prod_e d_e,$$

where the products are taken over the vertices v and edges e in Γ_{half} .

7.2. Normal bundles. For every flag $F = (v, e)$, let $j_F = j_v$. For every element $[f, \Sigma, (z_k, c(z_k))_k]$ in the fixed locus corresponding to Γ , there is an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Aut}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}} &\rightarrow \text{Def}(f)_{\mathbb{R}} \rightarrow \text{Def}(f, \Sigma, (z_k, c(z_k))_k)_{\mathbb{R}} \\ &\rightarrow \text{Def}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}} \rightarrow \text{Ob}(f)_{\mathbb{R}} \rightarrow \text{Ob}(f, \Sigma, (z_k, c(z_k))_k)_{\mathbb{R}} \rightarrow 0 \end{aligned}$$

Thus, as in [8, Section 27.6],

$$\begin{aligned} (7.1) \quad e(N_{\Gamma}^{\text{vir}}) &= e(\text{Def}(f, \Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}} / \text{Ob}(f, \Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}}) \\ &= \frac{e(\text{Def}(f)_{\mathbb{R}}^{\text{mov}} / \text{Ob}(f)_{\mathbb{R}}^{\text{mov}}) e(\text{Def}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}})}{e(\text{Aut}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}})}, \end{aligned}$$

where “mov” means the moving part (the part with the nonzero \mathbb{T} -weights) and $e(\cdot)$ denotes the equivariant Euler class. Following [8, Section 27.6], we now determine the three terms appearing on the right-hand side of (7.1).

For each edge e of Γ_{half} , $\text{Aut}(\Sigma, (z_k, c(z_k))_k)$ contains a \mathbb{T} -fixed one-dimensional complex subspace of infinitesimal automorphisms of the corresponding non-contracted component Σ_e which fix the two branch points of $f_e \equiv f|_{\Sigma_e}$; this subspace cancels with a similar piece in $\text{Def}(f_e)_{\mathbb{R}}$. In the separable case, the space $\text{Aut}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}$ also contains \mathbb{T} -fixed one-dimensional real subspaces of infinitesimal automorphisms of the central components Σ_{0j} ; this subspace cancels with a similar piece in $\text{Def}(f_{0j})_{\mathbb{R}}$, up to sign taken into account by Lemma [2, Lemma A.2]. The remaining automorphisms, none of which is \mathbb{T} -fixed, correspond to the vertices v in Γ_{half} of valence 1; they describe the infinitesimal automorphisms moving the branch point x_v of f_e , where e is the unique edge containing v , that lies over j_v . Thus, similarly to [8, Section 27.4],

$$(7.2) \quad e(\text{Aut}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}}) = \prod_{\substack{v \in e \\ \text{val}(v)=1}} e(T_{x_v} \Sigma_e) = \prod_{\substack{v \in e \\ \text{val}(v)=1}} w_{(v,e)},$$

where $w_{(v,\{v,v'\})} = \frac{\lambda_{j_v} - \lambda_{j_{v'}}}{d_{\{v,v'\}}}.$

A deformation of a contracted component of the domain (as a marked curve) is \mathbb{T} -fixed. The moving deformations come from smoothing (conjugate pairs) of nodes of Σ . For each node x of Σ inside Γ_{half} , $\text{Def}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}}$ contains the complex one-dimensional space isomorphic to the tensor product of the tangent spaces of the two components of Σ sharing x . There are two possibilities. Each $v \in \Gamma_{\text{half}}$ shared by two edges contributes $w_{F_1} + w_{F_2}$, where F_1 and F_2 are the two flags containing v . Each flag $F = (v, e)$ with $v \in \Gamma_{\text{half}}$ and $\text{val}(v) \geq 3$ contributes $w_F - \psi_F$, where $\psi_F \in H^2(\overline{\mathcal{M}}_v)$ is the first Chern class of the universal cotangent bundle on $\overline{\mathcal{M}}_v$ corresponding to the marked point determined by F on the contracted curve determined by the vertex v . Thus,

$$(7.3) \quad e(\text{Def}(\Sigma, (z_k, c(z_k))_k)_{\mathbb{R}}^{\text{mov}}) = \prod_{\substack{\text{val}(v)=2 \\ v \in e_1, e_2 \\ e_1 \neq e_2}} (w_{(v,e_1)} + w_{(v,e_2)}) \cdot \prod_{\text{val}(v) \geq 3} \prod_{v \in e} (w_{(v,e)} - \psi_{(v,e)}).$$

By the convention of Appendix A, in the non-separable case, the vertex $v = p_0$ and flags starting at this vertex are counted in the calculations over Γ_{half} , while \bar{p}_0 and corresponding flags are not.

In the separable case, there is an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Def}(f)_{\mathbb{R}} \rightarrow \oplus H^0(\Sigma_0, f_0^* T\mathbb{P}^{2m-1})_{\mathbb{R}} \oplus \bigoplus_{e \neq e_{0j}} H^0(\Sigma_e, f_e^* T\mathbb{P}^{2m-1}) \oplus \bigoplus_v T_{p_{j_v}} \mathbb{P}^{2m-1} \\ \rightarrow \bigoplus_F T_{p_{j_F}} \mathbb{P}^{2m-1} \rightarrow \text{Ob}(f)_{\mathbb{R}} \rightarrow \oplus_v H^1(\Sigma_v, f^* T\mathbb{P}^{2m-1}) \rightarrow 0, \end{aligned}$$

where the direct sums are taken over the vertices v , edges e , and flags F in Γ_{half} . Thus,

$$(7.4) \quad \frac{e(\text{Def}(f)_{\mathbb{R}}^{\text{mov}}/\text{Ob}(f)_{\mathbb{R}}^{\text{mov}}) = \prod_{e \neq e_{0j}} e(H^0(\Sigma_{e_0}, f_0^* T\mathbb{P}^{2m-1})_{\mathbb{R}}^{\text{mov}}) \prod_{e \neq e_{0j}} e(H^0(\Sigma_e, f_e^* T\mathbb{P}^{2m-1})^{\text{mov}}) \cdot \prod_v \prod_{j \neq j_v} (\lambda_{j_v} - \lambda_j)}{\prod_F \prod_{j \neq j_F} (\lambda_{j_F} - \lambda_j) e(H^1(\Sigma_v, f^* T\mathbb{P}^{2m-1})^{\text{mov}})}.$$

The contribution of $e \neq e_{0j}$ is standard and given by

$$(7.5) \quad e(H^0(\Sigma_e, f_e^* T\mathbb{P}^{2m-1})^{\text{mov}}) = (-1)^{d_e} \frac{d_e!^2}{d_e^{2d_e}} (\lambda_{j_1} - \lambda_{j_2})^{2d_e} \prod_{r=0}^{d_e} \prod_{k \neq j_1, j_2} \left(\frac{r\lambda_{j_1} + (d_e - r)\lambda_{j_2}}{d_e} - \lambda_k \right),$$

where j_1 and j_2 are the two vertex labels of the edge e ; see [8, Section 27.4]. The contribution of the half-edge e_{0j} is described by [2, Lemma A.2]. Finally, as in [8, Section 27.6], contribution of $e(H^1(\Sigma_v, f^* T\mathbb{P}^{2m-1})^{\text{mov}})$ is given by

$$\prod_v \prod_{j \neq j_v} c(\mathbb{E}^*) \left(\frac{1}{\lambda_j - \lambda_{j_v}} \right).$$

Here, $\mathbb{E} = H^0(\Sigma_v, \omega_{\Sigma_v})$ denotes the Hodge bundle of Σ_v . Since each Σ_v is rational, \mathbb{E} is null. Putting together we get (separable case)

$$(7.6) \quad \begin{aligned} 1/e(N_{\Gamma}^{\text{vir}}) &= \prod_{\substack{v \in e \\ \text{val}(v)=1}} w(v, e) \prod_{\substack{\text{val}(v)=2 \\ v \in e_1, e_2 \\ e_1 \neq e_2}} (w(v, e_1) + w(v, e_2))^{-1} \cdot \prod_{\text{val}(v) \geq 3} \prod_{v \in e} (w(v, e) - \psi_{(v, e)})^{-1} \\ &\quad \prod_v \prod_{j \neq j_v} (\lambda_{j_v} - \lambda_j)^{-1} \prod_F \prod_{j \neq j_F} (\lambda_{j_F} - \lambda_j) \\ &\quad \prod_{e \neq e_{0j}} (-1)^{d_e} \frac{d_e^{2d_e}}{d_e!^2} (\lambda_{j_1} - \lambda_{j_2})^{-2d_e} \prod_{r=0}^{d_e} \prod_{k \neq j_1, j_2} \left(\frac{r\lambda_{j_1} + (d_e - r)\lambda_{j_2}}{d_e} - \lambda_k \right)^{-1} \\ &\quad \prod_{k=1,2} \frac{1}{d_{0k}!} \left(\frac{d_{0k}}{2\lambda_i} \right)^{d_{0k}} \prod_{\substack{1 \leq j \leq 4m \\ j \neq i, 2|(i+j)}} \prod_{r=0}^{d_{0k}} \left(\frac{d_{0k} - 2r}{d_{0k}} \lambda_i - \lambda_j \right)^{-1} \end{aligned}$$

In the non-separable case, by the convention of Appendix A, there is an exact sequence of oriented vector spaces

$$(7.7) \quad \begin{aligned} 0 \rightarrow \text{Def}(f)_{\mathbb{R}} &\rightarrow \bigoplus_e H^0(\Sigma_e, f_e^* T\mathbb{P}^{2m-1}) \oplus \bigoplus_v T_{p_{j_v}} \mathbb{P}^{2m-1} \\ &\rightarrow \bigoplus_F T_{p_{j_F}} \mathbb{P}^{2m-1} \rightarrow \text{Ob}(f)_{\mathbb{R}} \rightarrow \oplus_v H^1(\Sigma_v, f^* T\mathbb{P}^{2m-1}) \rightarrow 0, \end{aligned}$$

where the direct sums are taken over the vertices v , edges e , and flags F in Γ_{half} . Thus,

$$(7.8) \quad e(\text{Def}(f)_{\mathbb{R}}^{\text{mov}}/\text{Ob}(f)_{\mathbb{R}}^{\text{mov}}) = \frac{\prod_e e(H^0(\Sigma_e, f_e^* T\mathbb{P}^{2m-1})^{\text{mov}}) \cdot \prod_v \prod_{j \neq j_v} (\lambda_{j_v} - \lambda_j)}{\prod_F \prod_{j \neq j_F} (\lambda_{j_F} - \lambda_j) e(H^1(\Sigma_v, f^* T\mathbb{P}^{2m-1})^{\text{mov}})}.$$

Putting together we get (non-separable case)

$$(7.9) \quad \begin{aligned} 1/e(N_{\Gamma}^{\text{vir}}) &= (-1) \prod_{\substack{v \in e \\ \text{val}(v)=1}} w_{(v,e)} \prod_{\substack{\text{val}(v)=2 \\ v \in e_1, e_2 \\ e_1 \neq e_2}} (w_{(v,e_1)} + w_{(v,e_2)})^{-1} \cdot \prod_{\text{val}(v) \geq 3} \prod_{v \in e} (w_{(v,e)} - \psi_{(v,e)})^{-1} \\ &\quad \prod_v \prod_{j \neq j_v} (\lambda_{j_v} - \lambda_j)^{-1} \prod_F \prod_{j \neq j_F} (\lambda_{j_F} - \lambda_j) \\ &\quad \prod_{e \neq e_{0j}} (-1)^{d_e} \frac{d_e^{2d_e}}{d_e!^2} (\lambda_{j_1} - \lambda_{j_2})^{-2d_e} \prod_{r=0}^{d_e} \prod_{k \neq j_1, j_2} \left(\frac{r\lambda_{j_1} + (d_e - r)\lambda_{j_2}}{d_e} - \lambda_k \right)^{-1}. \end{aligned}$$

The extra minus sign in (7.9) is due to the fact that the orientation given by Appendix A is the reverse of the orientation given by Section 5.

7.3. Genus one real invariants of \mathbb{P}^{2m-1} . From [2], we know that \mathbb{P}^{4m-1} has a canonical spin structure and $K_{\mathbb{P}^{4m-1}}$ has a canonical real square root. Therefore, the real genus one moduli spaces $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi}$ are oriented. Define

$$N_{1,d}^{\phi}(t_1, \dots, t_{\ell}) = \int_{[\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi}]^{\text{vir}}} \text{ev}_1^* H^{t_1} \wedge \dots \wedge \text{ev}_{\ell}^* H^{t_{\ell}}.$$

By the dimension formula (1.1), in order for this integral to make sense we should have

$$t_1 + \dots + t_{\ell} = \ell + 2md.$$

For example over \mathbb{P}^3 , we can consider $t_i = 3$ and define $N_{1,d}^{\phi} = N_{1,d}^{\phi}(3, \dots, 3)$. Similar numbers in genus zero are calculated for some low degrees in [2].

By the classical localization theorem of [1],

$$\begin{aligned}
(7.10) \quad N_d^\phi(t_1, \dots, t_\ell) &= \sum_{\Gamma \triangleright \mathbf{c}_a} \frac{1}{D_{\Gamma_{\text{half}}}} \int_{\mathcal{M}_{\Gamma_{\text{half}}}} \frac{\prod_{k=1}^{\ell} s_k^{t_k+1} \lambda_{j_{v(k)}}^{t_k}}{\text{Aut}(\Gamma) e(N_\Gamma)} \\
&\quad - \sum_{\Gamma \triangleright \mathbf{c}_m} \frac{1}{D_{\Gamma_{\text{half}}}} \int_{\mathcal{M}_{\Gamma_{\text{half}}}} \frac{\prod_{k=1}^{\ell} s_k^{t_k+1} \lambda_{j_{v(k)}}^{t_k}}{\text{Aut}(\Gamma) e(N_\Gamma)} \\
&\quad + \sum_{\Gamma \triangleright \mathbf{c}_k} \frac{1}{D_{\Gamma_{\text{half}}}} \int_{\mathcal{M}_{\Gamma_{\text{half}}}} \frac{\prod_{k=1}^{\ell} s_k^{t_k+1} \lambda_{j_{v(k)}}^{t_k}}{\text{Aut}(\Gamma) e(N_\Gamma)},
\end{aligned}$$

where the first and second sums are taken over the graphs Γ corresponding to the fixed loci in $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi, \mathbf{c}_a}$, $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi, \mathbf{c}_m}$, and $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi, \mathbf{c}_k}$ respectively. The negative sign arises due to the fact that we flip the orientation of $\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{4m-1}, [d])^{\phi, \mathbf{c}_m}$ when gluing it to the other two components.

Theorem 7.2. *For all $m, d, \ell, t_1, \dots, t_\ell \in \mathbb{Z}^+$,*

$$(7.11) \quad N_{1,d}^{\eta_{4m-1}}(t_1, \dots, t_\ell) = N_{1,d}^{\tau_{4m-1}}(t_1, \dots, t_\ell).$$

Furthermore, these invariants vanish if d is odd or $2|t_k$ for some k .

Proof. If $\phi = \tau_{2m-1}$ and d is odd, the involution on Σ should be separable and one of d_{0j} , say d_{01} , is odd and thus d_{02} is even. Therefore, Γ represents an element of either $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\tau_{2m-1}, \mathbf{c}_a}$ or $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\tau_{2m-1}, \mathbf{c}_m}$. Since the contribution of such Γ to the first and the second term are reverse of each other, the first two terms in (7.10) cancel out each other and last term is null; therefore, we get zero.

If $\phi = \eta_{2m-1}$, all moduli spaces except $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\eta_{2m-1}, \mathbf{c}_k}$ are empty. Moreover, d should be even and in the separable case both of d_{0j} should be odd.

Finally, if $\phi = \tau_{2m-1}$ and d is even, either $c = \mathbf{c}_k$ and the involution on Σ is non-separable or both of d_{0j} have the same parity. In the later case and if both d_{0j} are odd, Γ represents an element of $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\tau_{2m-1}, \mathbf{c}_a}$; otherwise, it does belong to all $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\tau_{2m-1}, \sigma}$, $\sigma = \mathbf{c}_a, \mathbf{c}_m, \mathbf{c}_k$. The contribution of the non-separable case and when both d_{0j} are odd is identical to the contribution of the same graph Γ_{half} , considered as an element of $(\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\eta_{2m-1}, \mathbf{c}_k})^{\mathbb{T}}$. In the remaining cases, for a given Γ_{half} , the contribution of Γ_{half} to the middle term is twice the contribution of Γ_{half} to the first and the third term in (7.10), because the corresponding maps in $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\eta_{2m-1}, \mathbf{c}_a}$ and $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\eta_{2m-1}, \mathbf{c}_k}$ have an extra \mathbb{Z}_2 automorphism group. Thus, the weighted sum of contribution of Γ_{half} to the three pieces of (7.10) cancel out each other. This proves the first and second claim of the theorem. Proof of the last

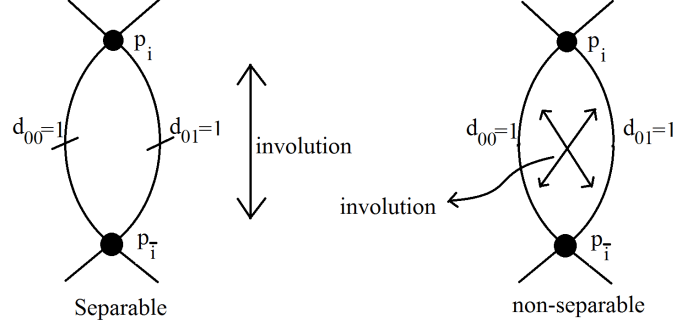


FIGURE 5. Possible degree 2 real graphs

claim on the vanishing of invariants if t_i is even is identical to the proof [2, Theorem 1.8]. \square

Example 7.3 ($d = 2, m = 1, \phi = \eta_3$). Together with the action of the involution, there are two possible cases, one separable and one non-separable, shown in Figure 5. Both have an automorphism group of order two which switches e_{01} and e_{02} . Applying the formula (7.6) to this graph (or equally to the half graph) we find the contribution of separable case is

$$4 \times \frac{1}{2} \left(\frac{\lambda_1^2}{8(\lambda_1^2 - \lambda_3^2)} + \frac{\lambda_3^2}{8(\lambda_3^2 - \lambda_1^2)} \right) = \frac{1}{4},$$

while the contribution of non-separable case is

$$4 \times \frac{1}{2} \left(\frac{-\lambda_1^2}{8(\lambda_1^2 - \lambda_3^2)} + \frac{-\lambda_3^2}{8(\lambda_3^2 - \lambda_1^2)} \right) = -\frac{1}{4};$$

therefore, $N_{1,2}^\phi = 0$. Note that every real degree 2 genus 1 map should be a double cover of some real line; thus, there are no such curves passing through two generic points in \mathbb{P}^3 .

Example 7.4. [$d = 4, m = 1, \phi = \eta_3$] In this case, ignoring the location of marked points, there are five possible graphs, three of them separable and two of them non-separable, shown in Figure 6. For each Γ let $\overset{\circ}{\Gamma}$ be the graph obtained by removing marked points; and let $e(\Gamma_{\text{half}})$ be the contribution of that calculated via (7.6) or (7.9). We use the following lemma to simplify the calculations.

Lemma 7.5. *Over the projective space \mathbb{P}^{4m-1} , if $t_1, \dots, t_\ell \equiv t$, the following identity holds.*

$$\frac{\prod_{k=1}^{\ell} \lambda_{j_{v(k)}}^{t_k}}{e(N_{\Gamma_{\text{half}}})} = \frac{\left(\sum_{F \in \overset{\circ}{\Gamma}_{\text{half}}} \frac{\lambda_{v(F)}^t}{w_F} \right)^\ell}{e(N_{\overset{\circ}{\Gamma}_{\text{half}}})}.$$

and \mathfrak{c}_k , we get

$$\begin{aligned}
 \text{contribution of } \mathfrak{c}_a - \text{type curves} &= \frac{12x^4 + 5x^2y^2 + 12y^4}{x^2y^2}, \\
 \text{contribution of } \mathfrak{c}_m - \text{type curves} &= -16 \frac{x^4 + x^2y^2 + y^4}{x^2y^2}, \\
 \text{contribution of } \mathfrak{c}_k - \text{type curves} &= 4 \frac{x^4 - x^2y^2 + y^4}{x^2y^2}.
 \end{aligned}
 \tag{7.12}$$

Thus, we conclude that each of $\mathcal{M}_{1,0,4}(\mathbb{P}^3, [4])^{\tau_3, c}$ have non-trivial boundaries. Therefore, contrary to the conjectures from physics (e.g. see [17]), individual \mathfrak{c}_a -type, \mathfrak{c}_m -type, and \mathfrak{c}_k -type open invariants may not always exist.

However, in the calculation of [17] and [13] over Calabi-Yau 3-fold complete intersections, each component of the moduli space contributes a weight-independent factor to the total sum; therefore, it might be true, in this case, that refined real invariants exist.

APPENDIX A. A DIFFERENT APPROACH FOR ORIENTING \mathfrak{c}_k -TYPE REAL MAPS

In this appendix, we will present another method for orienting $\mathcal{M}_{1,0,\ell}(X, A)^{\phi, \mathfrak{c}_k}$ which will be suitable in the localization calculation of non-separable curves.

Consider \mathbb{P}^1 with the coordinate chart $z \in \mathbb{C} \cup \{\infty\}$ and let $w = \frac{1}{z}$. Let \mathbb{P}_L^1 and \mathbb{P}_R^1 be two copies of \mathbb{P}^1 with coordinate charts z_L and z_R , respectively. Let

$$(A.1) \quad \Sigma_{\text{nod}} = \mathbb{P}_L^1 \cup \mathbb{P}_R^1 / 0_L \sim 0_R, \quad \infty_L \sim \infty_R,$$

be a 2-nodal curve obtained by identifying 0 and ∞ of the two components. There is a real structure on Σ_{nod} ,

$$\sigma_{\text{nod}}: z_L \rightarrow \bar{w}_R,$$

which sends the image of 0, say q_0 , to that of ∞ , say q_∞ , in Σ_{nod} , and thus has no fixed point. For $\epsilon \in \mathbb{R}^{>0}$, let Σ_ϵ be the genus one Riemann surface obtained by smoothing the two nodes of Σ via the gluing map

$$z_L z_R = \epsilon, \quad w_L w_R = \epsilon.$$

The involution σ_{nod} naturally extends to an involution σ_ϵ on Σ_ϵ with no fixed point, thus is of \mathfrak{c}_k -type. It is easy to show that every genus one real curve of \mathfrak{c}_k -type is isomorphic to a unique $(\Sigma_\epsilon, \sigma_\epsilon)$ for some $\epsilon > 0$. The two curves

$$(A.2) \quad C_0 = \{|z_L| = \sqrt{\epsilon} \subset \Sigma_\epsilon\}, \quad C_\infty = \{|w_L| = \sqrt{\epsilon} \subset \Sigma_\epsilon\},$$

are mapped to each other by the involution and generate the $+1$ -eigenspace of the action of \mathfrak{c}_k on $H_1(\Sigma)$.

With the notations as in Section 5, in the orientation problem for $\mathcal{M}_{1,0,\ell}(X, A)^{\phi, \mathfrak{c}_k}$, it is sufficient to consider the case $\ell = 0$. Similarly, for $[f] = [(\Sigma, \mathfrak{c}_k, j, u)] \in \mathcal{M}_{1,0,0}(X, A)^{\phi, \mathfrak{c}_k}$, in order to put an orientation on

$$T_{[f]} \mathcal{M}_{1,0,0}(X, A)^{\phi, \mathfrak{c}_k},$$

we should put an orientation on $T_f\mathcal{P}_{1,0,0}(X, A)^{\phi, \mathbf{c}_k}$ and $T_{\text{id}}G_{\mathbf{c}_k}$, whose quotient is invariant under the action of G_σ . We again have a decomposition

$$T_f\mathcal{P}_{1,0,0}(X, A)^{\phi, \mathbf{c}_k} = T_f\mathcal{P}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k} \oplus T_j\mathcal{M}_{\mathbf{c}_k}.$$

For $T_f\mathcal{P}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k}$, we need to consider the determinant of the index bundle

$$\text{ind}_{\mathbb{R}} E = \Lambda^{\text{top}} H^0(E)_{\mathbb{R}} \otimes \Lambda^{\text{top}} (H^1(E)_{\mathbb{R}})^*,$$

where $E = u^*TX$, and $H^0(E)_{\mathbb{R}}$ and $H^1(E)_{\mathbb{R}}$ are the kernel and cokernel of a real Cauchy-Riemann operator on E .

Let $(\Sigma, \mathbf{c}_k, j, u)$ be as above. By the argument above, we can assume $(\Sigma, \mathbf{c}_k, j) \cong (\Sigma_\epsilon, \sigma_\epsilon, j_\epsilon)$, for some $0 < \epsilon < 1$; this presentation is unique up to the action of $G_{\mathbf{c}_k}$. The map

$$\mathcal{M}_{\mathbf{c}_k} \rightarrow \mathbb{R}, \quad \Sigma_\epsilon \rightarrow \ln(\epsilon) \in \mathbb{R},$$

determines a parametrization and therefore an orientation on $T\mathcal{M}_\sigma$ (which is reverse of the orientation determined in Section 2). Fix one such presentation. Once such a presentation is fixed, it induces an orientation on $T_{\text{id}}G_{\mathbf{c}_k}$ via the isomorphism

$$T_{\text{id}}G_{\mathbf{c}_k} \cong \mathbb{R} \cdot v, \quad v = \frac{d}{d\theta}(e^{i\theta} z_L)|_{\theta=0}.$$

For δ very close to and smaller than 1, let A_0 and A_∞ be two small annuli around each of C_0 and C_∞ , respectively, given by

$$A_0 = \{\sqrt{\delta\epsilon} < |z_L| < \sqrt{\epsilon/\delta}\}, \quad A_\infty = \{\sqrt{\delta\epsilon} < |w_L| < \sqrt{\epsilon/\delta}\}.$$

By definition $\mathbf{c}_k(A_0) = A_\infty$. For every vector bundle E over Σ_ϵ , the restrictions $E_0 = E|_{A_0}$ and $E_\infty = E|_{A_\infty}$ are trivial but the choice of trivialization is not unique; fixing one choice, other choices of homotopy classes of trivialization are determined by $\pi_1(U(n)) = \mathbb{Z}$. Moreover, given trivialization

$$\psi_0: E_0 \rightarrow A_0 \times \mathbb{C}^n,$$

it canonically induces a trivialization on E_∞ given by

$$\psi_\infty = C \circ \psi_0 \circ d\phi,$$

where $C: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the complex conjugation.

Lets fix a choice of trivialization ψ_0 of E_0 and consider the associated trivialization ψ_∞ of E_∞ . Via the chosen trivializations, the bundle $(E, d\phi)$ descends to a bundle $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ over Σ_{nod} so that

$$d(\phi_{\text{nod}})|_{\mathbb{P}_L^1}: E_{\text{nod}}|_{\mathbb{P}_L^1} \rightarrow E_{\text{nod}}|_{\mathbb{P}_R^1}.$$

A section of $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ is of the form $\xi = (\xi_L, \xi_R)$, with matching conditions at the nodes. A section ξ is real if and only if

$$\xi_R(\sigma_{\text{nod}}(z)) = d(\phi_{\text{nod}})(\xi_L(z)), \quad \forall z \in \Sigma_L.$$

Therefore, it is determined by an arbitrary section of $E_{\text{nod}}|_{\mathbb{P}_R^1}$ with matching condition on the nodes. The matching condition at the nodes gives a short exact sequence

$$0 \rightarrow W^{1,p}(E_{\text{nod}})_{\mathbb{R}} \rightarrow W^{1,p}(E_{\text{nod}}|_{\mathbb{P}_L^1} \oplus E_{\text{nod}}|_{\mathbb{P}_R^1})_{\mathbb{R}} \rightarrow (\mathbb{C}_{q_0}^n \oplus \mathbb{C}_{\infty_0}^n)_{\mathbb{R}} \rightarrow 0.$$

We have

$$\begin{aligned} W^{1,p}(E_{\text{nod}}|_{\mathbb{P}_L^1} \oplus E_{\text{nod}}|_{\mathbb{P}_R^1})_{\mathbb{R}} &\cong W^{1,p}(E_{\text{nod}}|_{\mathbb{P}_L^1})_{\mathbb{R}} \\ (\mathbb{C}_{q_0}^n \oplus \mathbb{C}_{\infty_0}^n)_{\mathbb{R}} &\cong \mathbb{C}_{q_0}^n; \end{aligned}$$

thus, for the associated index of the pair $(E_{\text{nod}}, (d\phi)_{\text{nod}})$ there is an isomorphism

$$(A.3) \quad \text{ind}_{\mathbb{R}} E_{\text{nod}} \cong \text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\mathbb{P}_L^1}) \otimes_{\mathbb{C}} \det(\mathbb{C}_{q_0}^n)^*.$$

Each term on the right hand side is canonically oriented by its complex structure. We conclude that (A.3) induces an orientation on $\text{ind}_{\mathbb{R}} E_{\text{nod}}$. Then, a gluing argument analogous to that of [5, Proposition 8.1.4] determines an orientation on $\text{ind}_{\mathbb{R}} E$ itself.

Together with the provided orientation for $T_{\text{id}}G_{\mathbf{c}_k}$ and $T_j\mathcal{M}_{\mathbf{c}_k}$, this determines an orientation for $T_{[f]}\mathcal{M}_{1,0,0}(X, A)^{\phi, \mathbf{c}_k}$. It only remains to show that the provided orientation is independent of the presentation $(\Sigma, \mathbf{c}_k) \cong (\Sigma_{\epsilon}, \sigma_{\epsilon})$. The orientation on $T_j\mathcal{M}_{\mathbf{c}_k}$ is obviously independent of the presentation because any $h \in G_{\mathbf{c}_k}$ extends to identity on $\mathcal{M}_{\mathbf{c}_k}$. Thus, we need to show that the orientation constructed on the quotient

$$(A.4) \quad T_f\mathcal{P}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k} / T_{\text{id}}G_{\mathbf{c}_k}$$

is independent of the chosen parametrization. For this, we again need to show that induced action of **flip** and **sw** on (A.4) is orientation preserving.

Note that $\deg(E_{\text{nod}}|_{\mathbb{P}_L^1}) = \frac{c_1(A)}{2}$; therefore by Riemann-Roch theorem

$$h^0(E_{\text{nod}}|_{\mathbb{P}_L^1}) - h^1(E_{\text{nod}}|_{\mathbb{P}_L^1}) = \frac{c_1(A)}{2} + \dim_{\mathbb{C}} X.$$

Action of **flip** on Σ descends to Σ_{nod} , we still denote it by **flip**, such that **flip**: $\mathbb{P}_L^1 \rightarrow \mathbb{P}_R^1$ sends z_L to w_R . Similar to Lemma 5.2, the induced action of **flip** on $\text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\mathbb{P}_L^1})$ is anti-complex linear. The induced action of **flip** on $\det_{\mathbb{C}}(\mathbb{C}_{q_0}^n)^*$ is anti-complex linear as well. Considering the effect of **flip** on each individual term of (A.3), we realize that if $\dim_{\mathbb{C}} X$ is odd and $\frac{c_1(A)}{2}$ is even, then, the action of **flip** on $T\mathcal{P}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k}$ is orientation preserving. On the other hand, the action of **flip** on $T_{\text{id}}G_{\mathbf{c}_k}$ is also orientation preserving; therefore, under the assumption of Theorem 1.1, the constructed orientation on $T_{[f]}\mathcal{M}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k}$ is independent of the action of **flip**.

Similarly, the induced action of **sw** on $\text{ind}_{\mathbb{C}}(E_{\text{nod}}|_{\mathbb{P}_L^1})$ is anti-complex linear. The induced action of **sw** on $\det_{\mathbb{C}}(\mathbb{C}_{q_0}^n)^*$ is complex linear. Considering the effect of **sw** on each individual term of (A.3), we realize that if $\dim_{\mathbb{C}} X$ is odd and $\frac{c_1(A)}{2}$ is even, the action of **sw** on $T\mathcal{P}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k}$ is orientation reversing. On the other hand, the action of **sw** on $T_{\text{id}}G_{\mathbf{c}_k}$ is also orientation reversing; therefore, under the assumption of Theorem 1.1, the constructed orientation on $T_{[f]}\mathcal{M}_{1,0,0}(X, A, j)^{\phi, \mathbf{c}_k}$ is independent of the action of

sw. As in the previous case, the action of **sw** on $T_{\text{id}}G_\sigma$ is orientation reversing; therefore, under the assumption of Theorem 1.1, the orientation on $T_{[f]}\mathcal{M}_{1,0,0}(X, A, j)^{\phi, \sigma}$ constructed above is independent of the action of **sw**.

Putting together, we realize that under assumptions of Theorem 1.1, $T\mathcal{M}_{1,0,0}(X, A)^{\phi, \epsilon_k}$ and therefore $T\mathcal{M}_{1,0,\ell}(X, A)^{\phi, \epsilon_k}$ is orientable.

By the above method, the choice of orientation depends on the homotopy type of the choice of trivialization ψ_0 of E_0 . Let $\gamma \in \pi_1(U(n)) \cong \mathbb{Z}$ be a generator. For a given trivialization ψ_0 of E_0 , let $\gamma \cdot \psi_0$ be the trivialization twisted by γ .

Lemma A.1. *For a given trivialization ψ of E_0 , the induced orientation on $T\mathcal{M}_{1,0,\ell}(X, A)^{\phi, \epsilon_k}$ via ψ and $\gamma \cdot \psi$ are reverse of each other.*

Proof is similar to the proof of [5, Proposition 8.1.16].

Theorem A.2. *If (X^{2n}, ω, ϕ) is a simply-connected symplectic manifold with a real structure ϕ , n is odd, and $4|K_X(A)$ for every $A \in H_2(X)$, then every $\overline{\mathcal{M}}_{1,0,\ell}(X, A)^{\phi, \epsilon_k}$ is canonically orientable.*

Proof. Let C_0 be as in (A.2). Since X is simply connected, there is a disc D in X with $\partial D = C_0$. The restriction $TX|_D$ is trivial and every two trivialization are homotopic to each other. Therefore, a choice of D induces a choice of a homotopy class of trivialization ψ of E_0 ; this, together with the fact that $4|K_X$ determines an orientation on the moduli space. Now let D' be another such disc, and ψ' be the resulting trivialization. Then $\psi' \sim \gamma \cdot \psi$ for some $\gamma \in \pi_1(U(n))$. Let $B \in H_2(X)$ be the homology class obtained by connecting D and D' along their common boundary, since $K_X(B) = \text{image}(\gamma) \in \mathbb{Z} \cong \pi_1(U(n))$ is even, by Lemma A.1, the induced orientation by D and D' coincide. \square

A.1. Application to the projective space. For a non-separated element

$$f = [u, \Sigma, (z_k, c(z_k))_k] \in (\overline{\mathcal{M}}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi, \epsilon_k})^{\mathbb{T}}$$

as in Section 7, Σ has at least two conjugate nodes (p_0, \bar{p}_0) . Smoothing other nodes of Σ , we get a nodal curve which looks like Σ_{nod} of (A.1), where p_0 and \bar{p}_0 play the role of q_0 and q_∞ , respectively, and the half graphs correspond to \mathbb{P}_L^1 and P_R^1 , respectively. Thus, we can apply the orientating process above to $f = [u, \Sigma, (z_k, c(z_k))_k]$, where one of the half graphs play the role of \mathbb{P}_L^1 and p_0 play the role of q_0 in (A.3). Each term in (7.7) is canonically oriented via its complex structure, and all the contributions are standard. Putting together, we get (7.9), where the extra minus sign is canceling the fact that the orientation given by Appendix A on \mathcal{M}_{ϵ_k} is reverse the orientation given in Section 5. In order to finish the argument, we need to compare the orientations on $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi, \epsilon_k}$ given by Appendix A and Section 5. One might try to compare the two orientations in an abstract way, which is hard; but for the sake of calculations over the projective space, since $\mathcal{M}_{1,0,\ell}(\mathbb{P}^{2m-1}, [d])^{\phi, \epsilon_k}$ is connected, we may just compare the two orientations over some particular element. In fact, if we change the sign of (7.9), we will not get a weight-independent result.

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